Forecasting the Collapse of Speculative Bubbles: 
An Empirical Investigation of the S&P 500 Composite Index

ISMA Discussion Papers in Finance 2002-04
This version: March 2002

Chris Brooks
ISMA Centre, University of Reading, UK,

And

Apostolos Katsaris
ISMA Centre, University of Reading, UK

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Abstract

In this paper we test for the presence of periodically partially collapsing, positive and negative, speculative bubbles in the S&P 500 Composite Index for the period 1888-2001. We extend existing regime-switching models of speculative behaviour by including abnormal volume as an indicator of the probable time of the bubble collapse. Abnormal volume is included as both a classifying variable that helps predict the probability of the bubble surviving, and as a factor of risk in the surviving state equation. Increased volume is considered a signal that market beliefs concerning the future of the bubble are changing. We show that abnormal volume is a significant predictor and classifier of returns. Furthermore, we examine the financial usefulness of the augmented model by studying the risk-adjusted profits of a trading rule formed using inferences from it. Use of the augmented model trading rule leads to higher risk adjusted returns than those obtained from employing existing models or a buy and hold strategy.

Keywords: Stock market bubbles, fundamental values, dividends, regime switching, speculative bubble tests.

JEL Classifications: C51, C53, G12

Contacting Author(s):
Dr. Chris Brooks,
Reader in Financial Econometrics
ISMA Centre, University of Reading
Reading, RG6 6BA, UK
Tel: +44-(0)1189-316768
Fax:+44-(0)1189-314741
Email: c.brooks@ismacentre.rdg.ac.uk

Apostolos Katsaris,
ISMA Centre, University of Reading
Reading, RG6 6BA, UK
Tel: +44-(0)1189-318239
Fax:+44-(0)1189-314741
Email: a.katsaris@ismacentre.rdg.ac.uk

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1 INTRODUCTION

Equity markets have historically enjoyed periods of spectacular growth, especially in the late 1990’s. Such phenomenal price growth has caused the development of asset-pricing models that try to explain the evolution of prices as well as to produce profitable forecasts for investment decisions. Nevertheless, many of these models often fail to predict the movement of prices in financial markets accurately especially in periods of high volatility in asset returns. This failure is sometimes attributed to an inability of these models to capture ‘fundamental’ values accurately. Other researchers claim that in certain periods, fundamental values seem to be irrelevant in the pricing of financial assets. The behaviour of stock market prices prior to the 1929 and 1987 stock market crashes are the most frequently cited examples of such periods of fundamental value irrelevance (see for example Wanninski (1978), Shiller (2000), Camerer (1989), White (1990), Rappoport and White (1994), De Long and Shleifer (1991), Galbraith (1954)). The evolution of stock market prices in these two periods has inspired the search for other factors, beyond fundamentals, that might affect market prices and thus cause their ‘apparent’ deviations from fundamental values. One possible explanation for these deviations is the presence of speculative bubbles.

Rational speculative bubbles have intrigued financial theorists because they provide an alternative explanation of the evolution of prices that does not require the assumption of investor irrationality. Since Shiller’s (1981) seminal paper, several indirect and direct methodologies have been developed to test for the presence of bubbles in equity, currency and commodities markets and in monetary data. Indirect tests are based on the identification of bubbles through an examination of the distributional properties of actual prices (or returns) and fundamental values (see for example Blanchard and Watson (1982), Shiller (1981), LeRoy and Porter (1981), Mankiw, Romer and Shapiro (1985)), through tests for cointegrating relationships between actual prices and fundamental values (see Diba and Grossman (1988a), Hamilton (1986), Hamilton and Whiteman (1985), Campbell and Shiller (1987), Meese (1986), Hall, Psaradakis and Sola (1998)^1), or through an examination of the specifications of the present value relationship and the actual relationship between prices and dividends (see West (1987), Dezhbakhsh and Demirguc-Kunt (1990), Meese (1986)).

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^1 Hall, Psaradakis and Sola (1998) formulate a test that is able to identify periodically collapsing speculative bubbles and thus can be classified as a direct bubble test. Nevertheless, because it is based on the identification of periods of explosive behavior of macroeconomic data through a switching augmented Dickey Fuller test, we have classified it as part of the indirect tests of bubble presence.

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Since indirect tests are usually a joint test of bubble absence and of the validity of the present value model (see Kleidon (1986a,b), West (1988), Joerding (1988)) they are usually argued to provide only ‘hints’ for bubble presence and not proof. Direct tests by contrast examine the presence of specific forms of speculative bubbles by identifying the presence of bubble-like behaviour in financial and macroeconomic data. Direct tests have focused on the presence of deterministic bubbles (Flood and Garber (1980), Flood, Garber and Scott (1984), Salge (1997)), fads (Summers (1986), Fama and French (1988), Cutler, Potterba and Summers (1991)) and on the presence of periodically collapsing speculative bubbles (McQueen and Thorley (1994), Van Norden (1996), Van Norden and Schaller (1999), Bohl (2000), Sornette and Johansen (1997)).

Although direct tests find mixed evidence of speculative bubbles, all of these tests, with the exception of Sornette and Johansen (1997), assume that bubble crashes are exogenous events. However, it is logical to assume that bubble collapses are a result of a change in the beliefs of investors concerning the future of the bubble. In an important recent paper, Van Norden and Schaller (1999) (VNS hereafter) propose a switching regime speculative behaviour model for testing for the presence of periodically partially collapsing speculative bubbles. VNS construct a switching regime model of returns with a state independent probability of switching regimes. In their model the probability of switching regimes is a negative function of the absolute size of the bubble.

In this paper we augment the model of VNS for testing for the presence of periodically partially collapsing speculative bubbles, and include observed abnormal volume as a predictor of the time of the crash. We argue that abnormal volume is a predictor and a classifier of returns because it can provide information to investors about the belief of the market in the future of the bubble. Since all investors have access to information about past trading volume, upon the observation of increased volume in the market, they perceive that other investors are liquidating the bubbly asset and thus they may rush to liquidate their holdings. This will cause a bubble collapse. Collapses, in this case, are caused endogenously since more investor selling leads to an increased supply of the asset. We therefore incorporate abnormal volume in the VNS model and examine its forecasting ability for subsequent returns.

Furthermore, research on speculative bubbles has focused only on the problem of bubble identification and, to our knowledge, none of the papers in the literature has examined speculative bubbles as a base for the formulation of trading strategies that exploit knowledge about the implied probability of a stock market crash. In this paper we evaluate the usefulness of our augmented model and the VNS model by examining the profitability of trading rules formed on the basis of the probabilities of a crash and of a
rally in the next time period estimated from the models. We also employ a longer sample than VNS that is extended to include an extra 11 years of monthly data to January 2001, thus incorporating the large increases in the prices of equities that were experienced during the 1990’s.

The remainder of this paper is organised as follows. We will briefly describe the theory of periodically collapsing speculative bubbles and the switching regression framework developed by VNS to test for the presence of such bubbles in Section 2. In Section 3 we describe and derive the augmented switching bubble model. Section 4 presents the data and the methodology used to construct fundamental values and in Section 5 we present the results and tests of the robustness of the VNS and the augmented model. In Section 6 we examine the forecasting ability of our augmented model against the VNS model by looking at the profitability of trading strategies formed on the basis of inferences from these models. Section 7 concludes.

2 PERIODICALLY COLLAPSING SPECULATIVE BUBBLES

Under the assumptions of rational expectations, risk-neutrality, constant discount rates and market equilibrium, the no arbitrage condition must hold and thus the price of a stock must be given by the present value of its future cash flows:

\[ p_t = \frac{E_t(\frac{p_{t+1} + d_{t+1}}{1+i})}{(1+i)} \] (1)

In equation (1), \( p_t \) is the actual price of the stock at time \( t \), \( d_t \) is the dividend paid by the stock in period \( t \), \( E(\cdot) \) is the mathematical expectations operator, \( i \) is the constant expected real rate of return in equilibrium that is equal to the discount rate\(^2\). Equation (1) shows that the actual price of a stock must be equal to the present value of the expected future price, at which the security will be sold, plus the expected dividend the investor will receive at time \( t+1 \).

If we allow for multi-period horizons then equation (1) can be updated with the expectation for the price at \( t+1 \) and substituted it into the original equation. Substituting recursively for all future prices in an

\(^2\) Fama and French state that the expected or required return of an asset is the discount rate that relates a present value with its future cash flows (Fama and French (1988))
infinite planning horizon and using the property of rational expectations $E_t[E_{t+1}(s_{t+2})] = E_t(s_{t+2})$ will yield the fundamental value of a stock:

$$p_t^f = \sum_{g=1}^{\infty} \frac{1}{(1+i)^g} E_t(d_{t+g})$$

Equation (2) describes the fundamental price of a stock ($p_t^f$) under an infinite planning horizon. If we allow the actual market price of the stock to deviate from its fundamental value then the actual stock-price can be described by the relationship:

$$p_t^a = p_t^f + b_t + u_t$$

where $b_t$ is the bubble component at period $t$, and $u_t$ is a zero mean, constant variance error term that contains the unexpected innovation of both the bubble term and of the fundamental component. The bubble component ($b_t$) is simply the difference between the actual price and the fundamental price of the security and is assumed here to have an evolutionary process that causes the systematic divergence of actual prices from their fundamental values. This bubble deviation can have different generating processes depending on the type of bubble that is included in the stock price. There are several types of bubbles that have different characteristics, different evolutionary processes and are generated by different factors. Nevertheless, all bubble processes must satisfy the no arbitrage condition since the price of the security must satisfy (1).

Blanchard (1979), and Blanchard and Watson (1982) formulate a speculative bubble model in which the bubble component continues to grow with explosive expectations in the next time period with probability $q$, or crashes to zero with probability $1-q$. If the bubble collapses then the actual price will be equal to the asset’s fundamental value. In their model, the explosive behaviour of bubble returns compensates the investor for the increased risk of a bubble crash as the bubble grows in size. According to Blanchard and Watson (1982), the expected bubble in period $t+1$ will be generated by the following stochastic process:

$$E_t(b_{t+1}) = \begin{cases} 
\frac{(1+i)b_t}{q} & \text{with probability } q(B_t) \\
0 & \text{with probability } 1-q(B_t) 
\end{cases}$$

where $q$ is the probability that the bubble will continue to exist in period $t+1$, ($0<q<1$). Equation (4) shows that if the bubble does not burst in period $t+1$, then the expected bubble must grow at a rate higher
than \( i \) in order to compensate the investor for the probability of collapse. On the other hand, the bubble might burst in period \( t+1 \) with probability \( 1-q \) and so the expected bubble will be equal to zero.

From (4) we note that if the bubble term at period \( t \) crashes to zero then it cannot regenerate since the expected bubble is equal to zero. This implies that there can be only one observed bubble in any financial time series (Diba and Grossman (1988a)). Furthermore, in (4) it is assumed that the bubble crashes immediately to its collapsing state value. These are restrictive assumptions, since it is plausible that there could be several bubble episodes in a financial time series or that a bubble could slowly deflate for several time periods or it might stop growing and remain at an approximately constant level for some time and then collapse or start growing again. In 1929 and 1987, the market peaked in late August and crashed in October whereas the strong correction in the Tokyo Stock Exchange took several months after January 1990.

Moreover, the explosive nature of bubbles leads Diba and Grossman (1988b) to conclude that under rational expectations, negative bubbles cannot exist since investors cannot rationally expect the value of a stock to become negative in finite time. This arises since if a negative rational speculative bubble exists, the bubble will grow geometrically causing the stock price to decrease without bound and become negative in finite time. However, Blanchard and Fischer (1989) claim that the arguments against the possibility of negative bubbles rely on a very strict form of rationality. Although the probability that the stock price will became zero or arbitrarily large (depending on the form of the bubble) is positive, this probability might be too small or the event may happen in the too distant future and thus investors decide to ignore it\(^3\). Finally, in the original model of Blanchard (1979) and Blanchard and Watson (1982), the probability of the bubble continuing to exist is non-observable and assumed constant.

In order to lift the requirement for these unrealistic restrictions, VNS formulate a periodically, partially-collapsing, positive and negative speculative bubble model that has a time varying probability of collapse. VNS consider the following bubble process:

\[
E_t(b_{t+1}) = \frac{(1+i)B_t}{q(B_t)} - \frac{1-q(B_t)}{q(B_t)} u(B_t) p_t^a \quad \text{with probability } q(B_t)
\]

\[
E_t(b_{t+1}) = u(B_t) p_t^a \quad \text{with probability } 1-q(B_t)
\]

\(^3\) See Weil (1990) on the possibility of price decreasing bubbles.
In (5), $B_t$ is the size of the bubble relative to the actual price $p^*_t$, $(B_t = b_t / p^*_t)$. $u(B_t)$ is a continuous and everywhere differentiable function such that: $u(0) = 0$ and $0 \leq \partial u(B_t)/\partial B_t < 1$, $q(B_t)$ is the probability of the bubble continuing to exist that is a negative function of the absolute relative size of the bubble.

As noted by Kindleberger (1989), a crash becomes more likely as the relative size of the bubble becomes larger. To incorporate this and to insure that the estimates of the probability of survival are bounded between zero and one, VNS employ a probit specification and allow the probability of survival to depend on the relative absolute size of the bubble, similar to the specification used by Bollerslev and Hodrick (1995):

$$q(B_t) = \Omega(\beta_{q,0} + \beta_{q,b} |B_t|)$$

(6)

where: $\Omega$ is the standard normal cumulative density function, $\Omega(\beta_{q,0})$ is the mean probability of being in the collapsing regime and $\beta_{q,b}$ is the sensitivity of the probability of collapse to the absolute relative size of the bubble. In equation (6), VNS allow for negative speculative bubbles by modelling the probability of the bubble surviving as a function of the absolute size of the bubble.

VNS also allow for partial bubble collapses by letting the expected bubble size in the collapsing state be a function of the relative bubble size. Under this setting the expected relative bubble size of period $t+1$ in the collapsing state must be smaller than the expected bubble relative size in the surviving state and not bigger than the bubble in period $t$. The assumption of a continuous and everywhere differentiable function is required so that they are able to linearise the model. This functional form is not imposed on the data but is required in order to derive empirically testable implications from the bubble model.

The VNS model can be used in order to specify asset returns as state dependent, where the state is unobservable. This implies that the security’s gross returns are given, under certain assumptions about the dividend process, by the following non-linear switching model:

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4 Evans (1991) and Hall and Sola (1993) also consider partial bubble collapses.


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where \( r_{t+1}^S \) denotes the gross return of period \( t+1 \) conditioning on the fact that the state at time \( t \) is the survival state \( (S) \) or the collapsing state \( (C) \) and on all other available information at time \( t \), and \( M \) is the gross fundamental return on the security. In order to estimate the model, VNS linearise equations (7) and (8) and derive a linear regime switching model for gross stock market returns with a single state-independent probability of switching regimes \( q(B_t) \):

\[
\begin{align*}
E(r_{t+1} \mid S) &= \left[ M (1 - B_t) + \frac{MB_t}{q(B_t)} - \frac{1 - q(B_t)}{q(B_t)} u(B_t) \right] & \text{with probability } q(B_t) \\
E(r_{t+1} \mid C) &= \left[ M (1 - B_t) + u(B_t) \right] & \text{with probability } 1 - q(B_t)
\end{align*}
\]

(7)

(8)

In (9), \( u_{S,t+1}^S \) and \( u_{C,t+1}^C \) are the unexpected gross returns of period \( t+1 \) in the surviving and in the collapsing regime respectively and are assumed zero mean and constant variance i.i.d. normal random variables. VNS estimate the above three equations with maximum likelihood, using the value weighted index of all stocks from the Center for Research on Security Prices (CRSP) database for the period January 1926 to December 1989. VNS find that there is non-linear predictability in stock market returns and that the deviations of actual prices from fundamental values are a significant factor in predicting both the level and the generating state of returns. Furthermore, from the switching speculative bubble model, VNS derive conditional probabilities of a crash and of a rally in the next time period and find that their model has explanatory power over several periods of apparently speculative behaviour of the data, although some of the observed crashes can be explained better by a model of regime switching in fundamental values. In what follows, we will re-estimate the VNS model using a larger sample on the S&P 500 Composite Index, in order to directly compare the results of this model with our augmented version.

3 AN AUGMENTED MODEL FOR PERIODICALLY COLLAPSING SPECULATIVE BUBBLES

In most of the speculative bubble models, and in the VNS model, a bubble collapse is a random event that may or may not be fuelled by the arrival of news. When the collapse occurs without the arrival of ‘bad’ news, speculative bubble models implicitly assume that investors randomly organise and decide to sell at
the same time thus causing the bubble to collapse. This collapse may be sharp if investors all sell at once or smooth if the market is more balanced between buyers and sellers. In the VNS model, the probability of a crash is a function of only the size of the bubble. Although investors can observe the probability that the bubble will collapse, they still face uncertainty about the time of collapse. It is, however, logical to assume that investors examine other variables in order to identify the optimal time of exit from the market.

In order to overcome this problem, we assume that investors try to time the collapse of a bubble by drawing inferences about the trading strategies of other investors from non-price variables. Our main suggestion is that bubble collapses occur because investors observe a signal that leads them to the conclusion that the market is no longer expecting the bubble to continue to exist. Once they observe this signal they start to liquidate their holdings and thus cause the bubble to collapse. We consider abnormal trading volume as a possible signal of the bubble collapsing.

The relationship between trading volume and stock returns has been extensively researched in the literature (see Karpoff (1987) for a meticulous survey of the literature until 1987). Ying (1966) finds that large increases in volume are usually followed by large stock market movements. Using monthly data, Morgan (1976) found that the variance of returns is positively correlated to volume and claims that volume is associated with systematic risk. This is supported by his findings that increased volume has a contemporaneous positive relationship with the change in price.

There are several theories that try to explain the relationships between volume and price, volume and returns, volume and skewness and volume and kurtosis. One theory that has been extensively discussed in the literature is that trading volume is a proxy for the degree of disagreement in the stock market (the reader is referred to Varian (1989), Harris and Raviv (1993), Kandel and Pearson (1995) and Odean (1998) for other models with the same feature). Although Karpoff (1986) claims that abnormal volume is not proof of investor disagreement (either ex-ante or ex-post) about information that is available, Shalen (1993) claims that volume and return volatility have a positive relationship with the ex ante dispersion of expectations about future prices.

He and Wang (1995) examine trading volume in a speculative market in which investors have differential information about the fundamental value of a stock and claim that when uncertainty is high the observed volume in the market is high. Marsh and Wagner (2000) examine the relationship between above average trading activity and conditional return variance and find that abnormal volume can significantly explain
volatility persistence but that this relationship is weaker when they consider extreme returns. Nevertheless, they conclude that, especially for the U.S. stock market, abnormal volume can help explain both the negative and the positive tails of the returns distribution but the effect is symmetric. However, Hong and Stein (1999) and Chen, Hong and Stein (2001) claim that divergence of information, approximated by abnormal trading volume, causes negative skewness in stock market returns.

Based on the above, we can conclude that abnormally high trading volume could be viewed as a signal of changing market expectations about the future of a speculative bubble. We suggest that abnormal volume signals an increase in the probability of a bubble collapsing, thus implying an increase in the probability of observing a negative (positive) return if a positive (negative) speculative bubble is present. This is the main difference between our model and the models of price – volume relationship described above. These models state that volume increases both tails of the distribution of expected returns or it signifies a decrease in the skewness of future returns. We consider abnormal volume as a sign that other investors are selling the bubbly asset.

For abnormal volume to signal an imminent bubble collapse would require the assumption that investors have different endowments implying that there are agents in the economy that do not hold equity and may decide to do so at a future date. The effect would be the same if the number of investors changes over time. Furthermore, we assume that some investors face short selling constraints and that in the short run, the supply of equity from firms, through IPOs and SEOs is limited. The latter assumption combined with the unwillingness of investors to sell the bubbly asset because of the expectation of high returns in the surviving regime, cause the supply curve to be relatively inelastic. Under this setting, speculative bubbles are a form of ‘demand side inflation’ in stock prices.

As the bubble continues to grow, the probability of a crash increases and thus some investors will decide to liquidate their holdings for profit taking or because they perceive a crash to be imminent. If a

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6 Note that all of the theories on the relationship between prices and volume are based on the asymmetric information of investors, on the heterogeneity of expectations or on the heterogeneity of investors’ preferences to risk. In our model, however, we assume that all investors have homogeneous information and are risk neutral. Both of these assumptions are necessary in order to estimate the bubble deviations since constant discount rates and common beliefs about the fundamental value are assumed. If investors are risk averse or have different degrees of risk aversion then the fundamental values estimated will be different for every investor since the discount rate will be different. If investors have heterogeneous information then they will perceive different fundamental values and thus different bubble sizes.
sufficiently large number of investors decide to sell the bubbly asset, supply will increase significantly and thus volume will be abnormally high while the rate of increase in prices will slow. This abnormally high volume will signal that a bubble collapse is imminent. From the above, we conclude that investors try to estimate the time of the crash by examining abnormal volume. Note that abnormal volume affects directly the probability of a crash and indirectly the expected return of the asset. The model for periodically collapsing speculative bubbles we consider is thus:

\[
E_t(b_{t+i}) = \frac{(1+i)b_t}{q(B_t, V_t^>)} - \frac{1 - q(B_t, V_t^>)}{q(B_t, V_t^>)} u(B_t) P_t \quad \text{with probability } q(B_t, V_t^>)
\]

\[
\text{with probability } 1 - q(B_t, V_t^>)
\]

In (10) \(q(B_t, V_t^>)\) is the probability of the bubble continuing to exist that is a function of the relative absolute size of the bubble and the measure of abnormal volume where: \(\partial q(B_t, V_t^>) / \partial |B_t| < 0\) and \(\partial q(B_t, V_t^>) / \partial V_t^> < 0\), and \(V_t^>\) is a measure of unusual volume in period \(t\). Following VNS, in equation (10) the expected size of the bubble is a function of the probability of a crash, the size of the bubble at period \(t\) and the function \(u(B_t)\) which is the relative size of the bubble in the collapsing state. The probability of the bubble continuing to exist is a negative function of the absolute (since we allow negative bubbles to exist) size of the bubble, and the measure for abnormal volume. From (12) we can derive the gross returns on the stock when a bubble is present under the assumption that dividends follow a geometric random walk with a constant drift. Under this assumption we can show that the expected next period gross returns are given by:

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7 Constrained investors are usually mutual funds, whose charters prohibit short positions, or small individual investors, and unconstrained investors can be hedge funds or other arbitrageurs.

8 Theories of herding and mimetic contagion in the stock market claim that when investors face incomplete information they try to improve their information set by looking to the market for hints about the trading strategies of other investors (Topol (1991), Kirman (1993), Lux (1995), Sornette and Johansen (1997)). Although this trading strategy could be considered naïve, since uninformed investors do not take into account the fact that price changes can be caused by the trading strategies of other uninformed investors, this behaviour can be rational if investors do not have other sources of information (Lux (1995), Orlean (1989)).

9 This is a common assumption in the literature; see for example Fama and French (1988), Kleidon (1986a).

10 Proof of these equations is not presented here in the interest of brevity but is available in an appendix upon request from the authors.
\[ E(r_{t+1} | S) = M(1 - B_t) + \frac{M}{q(B_t, V_t^*)} B_t - \frac{1 - q(B_t, V_t^*)}{q(B_t, V_t^*)} g(B_t) \] (11)

\[ E(r_{t+1} | C) = [M(1 - B_t) + g(B_t)] \] (12)

\[ P(r_{t+1} | S) = q(B_t, V_t^*) = \Omega(\beta_{g,0} + \beta_{g,b} B_t) + \gamma_{q,v} V_t^* \] (13)

where: \( \gamma_{q,v} \) is the sensitivity of the probability of survival to the measure of abnormal volume. Note that the measure of abnormal volume affects the expected returns on the asset only indirectly through the probability of the process being in state \( S \) or \( C \). Abnormal volume is thus suggested to signal an increase in the size of the tails of the distribution of expected returns that would signify a higher probability of a sharp collapse in the bubble. We follow VNS and select a probit model for the probability of survival \( P(r_{t+1} | S) \) since it satisfies the conditions set above and it ensures that probability estimates are bounded between zero and one. Equations (11), (12) and (13) are non-linear and very difficult to estimate. In order to estimate these equations, we linearise them by taking the first order Taylor series approximation of the model around an arbitrary \( B_0 \) and \( V_0^* \) and arrive at a linear switching regression model:

\[ r_{t+1} = \beta_{S,0} + \beta_{S,b} B_t + \beta_{S,v} V_t^* + u_{r_{t+1}}^S \] (14)

\[ r_{t+1} = \beta_{C,0} + \beta_{C,b} B_t + u_{r_{t+1}}^C \] (15)

\[ P(r_{t+1} | S) = q(B_t, V_t^*) = \Omega(\beta_{g,0} + \beta_{g,b} |B_t|) + \gamma_{q,v} V_t^* \] (16)

where: \( u_{r_{t+1}}^S \) is the unexpected gross return in the surviving regime and \( u_{r_{t+1}}^C \) is the unexpected gross return in the collapsing regime. Equation (14) states that the returns in the surviving regime are a function of the relative size of the bubble and of the measure of abnormal volume. In effect, equation (14) implies that as the bubble grows, investors demand higher returns in order to compensate them for the probability of a bubble collapse and since abnormal volume signals a possible change in the long run trend in equity prices, investors want to be compensated for this risk as well. The above linear switching regression model has a single state-independent probability of switching regimes \( P(r_{t+1} | S) \) that is a function of the relative size of the bubble and of the measure of abnormal volume.

We estimate the augmented model under the assumption of disturbance normality using maximum likelihood to directly maximise the following log-likelihood function:
\[
\ell (r_{t+1} | \xi) = \sum_{i=1}^{T} \ln \left[ P(r_{t+1} | S) \left( \frac{r_{t+1} - \beta_{S,0} - \beta_{S,b} B_t - \beta_{S,V} V_t}{\sigma_S} \right) + P(r_{t+1} | C) \left( \frac{r_{t+1} - \beta_{C,0} - \beta_{C,b} B_t}{\sigma_C} \right) \right]
\]

where: \( \xi \) is the set of parameters over which we maximize the likelihood function which includes \( \beta_{S,0}, \beta_{S,b}, \beta_{S,V}, \beta_{C,0}, \beta_{C,b}, \beta_{q,b}, \sigma_S, \sigma_C, \omega \) is the standard normal probability density function (pdf), \( \sigma_S, \sigma_C \) is the standard deviation of the disturbances in the surviving (collapsing) regime, and \( P(r_{t+1} | C) = 1 - P(r_{t+1} | S) \).

Note that the maximization of this log-likelihood function produces consistent and efficient estimates of the parameters in \( \xi \), as it does not require any assumptions about which regime generated a given observation. The above model is similar to the models described in Goldfeld, and Quandt (1976). We estimate both the VNS model and our augmented model in Matlab 5.3 using the BFGS algorithm.

From the first order Taylor series expansion we can derive certain conditions that must hold if the periodically collapsing speculative bubble model has explanatory power over stock market returns. If the above model can explain the variation in future returns then this would be evidence in favour of the presence of periodically collapsing speculative bubbles in the data. These restrictions are:

\begin{align*}
\beta_{S,0} &\neq \beta_{C,0} \quad \text{(a)} \\
\beta_{C,b} &< 0 \quad \text{(b)} \\
\beta_{S,b} &> \beta_{C,b} \quad \text{(c)} \\
\beta_{q,b} &< 0 \quad \text{(d)} \\
\gamma_{q,v} &< 0 \quad \text{(e)} \\
\beta_{S,V} &> 0 \quad \text{(f)}
\end{align*}

Restriction (a) implies that the mean return across the two regimes is different, so that there exist two distinct regimes, although we cannot say anything about the relative size of these coefficients. Restriction (b) implies that as the bubble grows, the expected return, if the collapsing regime is observed, should be negative. This means that the bubble must be smaller in the following period if the bubble collapses. Note that the opposite holds for negative bubbles: the larger the negative bubble the more positive the returns in the collapsing regime. Restriction (c) ensures that the bubble yields higher (lower) returns if a positive (negative) bubble is observed in the surviving regime than in the collapsing regime. Finally, restriction (d)
must hold by construction since the probability of the bubble continuing to exist is expected to decrease as the size of the bubble increases. Restrictions (a) to (d) are equivalent to the restrictions derived by VNS. The additional restriction (e) must hold so that abnormally high volume signals an imminent collapse of the bubble. Restriction (f) states that the coefficient on the abnormal volume measure must be greater than zero since, as volume increases investors perceive an increase in market risk.

We examine the power of the model to capture bubble effects in the returns of the S&P 500 by testing the model against three simpler specifications that capture stylised features of stock market returns. These models are nested within the speculative bubble model. We repeat these tests and also examine the augmented model against the simpler VNS model using a likelihood ratio test. Firstly, we examine whether the effects captured by the switching model can be explained by a more parsimonious model of changing volatility. In order to test this alternative, we follow VNS and jointly impose the following restrictions:

\[
\begin{align*}
\beta_{s,0} &= \beta_{c,0} \quad (18.1) \\
\beta_{s,b} &= \beta_{c,b} = \beta_{q,b} = 0 \quad (18.2) \\
\sigma_u^s &= \sigma_u^c \quad (18.3)
\end{align*}
\]

Restriction (18.1) implies that the mean return across the two regimes is the same and restriction (18.2) states the bubble deviation has no explanatory power for next period returns or for the probability of switching regimes. The later point suggests that there is a constant probability of switching between a high variance and low variance regime as this is stated in restriction (18.3).

In order to separate restrictions (18.1) and (18.2), we examine whether returns can be characterised by a simple mixture of normal distributions model, which only allows mean returns and variances to differ across the two regimes. This mixture of normal distributions model implies the following restrictions:

\[
\beta_{s,b} = \beta_{c,b} = \beta_{q,b} = 0 \quad (19)
\]

Another possible alternative is that of mean reversion in prices (fads) as described by Cutler, Poterba and Summers (1991). Under the fads model, returns are linearly predictable although mean returns do not differ across regimes. Furthermore, the deviation of actual prices from the fundamentals has no predictive ability over the probability of switching regimes. The returns in the two regimes are characterised by different variances of residuals but are the same linear functions of bubble deviations. The fads model is thus:
As a final statistical test, we also examine the robustness of our model against the more parsimonious VNS model by testing whether abnormal volume should be included in the speculative bubble model. The restrictions of this last test are:

$$\beta_{S,V} = \gamma_{q,V} = 0$$

(21)

The results of the speculative bubble models and the implied restrictions are discussed in Section 5.

4 DATA AND FUNDAMENTAL VALUES

The data that employed to test for the presence of bubbles comprise 1357 monthly observations on the S&P 500 for the period January 1888 – January 2001. The S&P 500 prices and dividends that are used to calculate fundamental prices and gross returns are taken from Shiller\textsuperscript{11}. We calculate monthly share volume by taking the sum of daily share volume reported by the NYSE for the period January 1888 – January 2001\textsuperscript{12}. Unusual trading volume is defined as the percentage deviation of last month’s volume from the 6 month moving average\textsuperscript{13}. The monthly dividend and price series are transformed into real variables using the monthly U.S. Consumer Price - All Items Seasonally Adjusted Index reported in Shiller.

In order to construct fundamental values, we examine two different specifications that use only information on past prices and dividends. The models used are the dividend multiple model of Schaller

\textsuperscript{11} Data available at: http://www.econ.yale.edu/~shiller/data.htm. For a description of the data used see also Shiller (1999) and the description online. Shiller’s sample ends in January 2000. For this reason we update his sample until January 2001 using data obtained from Datastream. In order to verify that the two datasets are consistent with each other we compare Shiller’s data from January 1965 to January 2000 with the values from DataStream and find no differences.

\textsuperscript{12} Data available at: http://www.nyse.com/marketinfo/marketinfo.html.

\textsuperscript{13} We also examined unusual trading volume measures using 3, 12 and 18 month moving averages but found that the deviation from the 6-month moving average has the highest explanatory power in predicting both the level and
and Van Norden (1999), and a mathematical manipulation of Campbell and Shiller’s (1987) VAR model of dividend components of prices. The dividend multiple model of fundamental values assumes a constant dividend–price ratio whereas the Campbell and Shiller measure allows for predictable variation in the dividend growth rate. Both models assume constant discount rates.

### 4.1 DIVIDEND MULTIPLE MEASURE OF FUNDAMENTALS

Schaller and van Norden (1999) show that if the discount rate is constant, then stock market prices follow the period-by-period arbitrage condition:

\[
p_t = E_t(p_{t+1} + d_{t+1})
\]

Assuming that dividends follow a geometric random walk, i.e. that log dividends follow a random walk with a drift, it can be shown that the fundamental price of a stock will be equal to a multiple of current dividends:

\[
p_t' = \rho d_t
\]

where: \( \rho = \frac{1 + r}{(1+\sigma^2/2)} - 1 \)

We use the sample mean of the price-dividend ratio to approximate \( \rho \). In this setting, the relative bubble size is given by:

\[
B_t = \frac{b_t}{p_t} = \frac{p_t - p_t'}{p_t} = 1 - \frac{\rho d_t}{p_t}
\]

### 4.2 CAMPBELL AND SHILLER FUNDAMENTAL VALUES

The dividend multiple measure of fundamental values assumes that the expected dividend growth rate is constant. In order to allow for predictable variation in the dividend growth rate, we estimate fundamental values based on the Campbell and Shiller (1987) dividend component of prices. It can be shown that the

the generating state of returns. The results for the other measures of abnormal volume are not presented for brevity and are available upon request from the authors.

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spread between stock prices and a constant multiple of current dividends is the optimal forecast of a multiple of the discounted value of all future dividend changes:

$$S_t = p_t^0 - \frac{1+i}{i} d_t = \frac{1+i}{i} \left( 1 + \sum_{g=1}^{\infty} \frac{1}{(1+i)^g} E_t[\Delta d_{t+g}] \right)$$

(25)

Using the VAR methodology created by Campbell and Shiller, we examine whether changes in dividends can be forecasted by the spread between prices and the multiple of current dividends. If the changes in dividends cannot be forecasted by the spread, this would imply that investors use only past dividends to form expectations about future dividends. If, on the other hand, investors include other variables in their information set then this information will be reflected in past prices and thus past realisations of the spread. This would imply that the spread has power to forecast future dividend changes.

We examine this relationship using the following VAR:

$$\begin{bmatrix} \Delta d_t \\ S_t \end{bmatrix} = \begin{bmatrix} a(L) & b(L) \\ c(L) & d(L) \end{bmatrix} \begin{bmatrix} \Delta d_{t-1} \\ S_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

(26)

where: both $\Delta d_t$ and $S_t$ are de-meaned. This matrix equation can be rewritten more compactly as:

$$z_t = Az_{t-1} + v_t$$

(27)

This VAR can be used to forecast future dividend changes conditional on the information set ($H_t$) described above that contains data on past dividend growth rates and realisations of the spread. Since:

$$E_t(z_{t+1}|H_t) = A'z_t$$

(28)

then the present value of future dividend changes, equal to the fundamental spread, can be forecasted from equation (28) using the following equation:

$$S_t^* = E_t(S_t^f|H_{t-1}) = \left( \frac{1+i}{i} \right) \omega^f \left( \frac{1}{1+i} \right) A \left( I - \left( \frac{1}{1+i} \right) A \right)^{-1} z_t$$

(29)

where $\omega$ is a row vector that picks out $\Delta d_{t-1}$. The fundamental price can thus be calculated by solving equation (25):

---

See Campbell and Shiller (1987) for more details concerning the methodology.
where \( i \) is the average real total return over the entire period. We construct fundamental values using equation (30). The second measure of relative bubble size is thus given by:

\[
B_t = 1 - \frac{S_t^* + \left( \frac{1+i}{i} \right) d_{t-1}}{p_t} \tag{31}
\]

Figure 1 presents the bubble deviations calculated from equation (24) and (31) for the entire sample. Note that both bubble deviations are increasingly large in 1929, 1987 and the late 1990’s. The bubble deviations are significantly negative in 1917, 1932, 1938, 1942 and 1982. The Campbell and Shiller measure of bubble deviations (dotted line) displays significantly more short term variability whereas the dividend multiple measure has larger and more persistent broad swings.

5 THE RESULTS

The results of the augmented model of speculative behaviour, for the dividend multiple measure of fundamental values, are presented in the first panel of Table 1 alongside the results of the VNS model for comparison. The second panel of the table contains the results of the LR tests of the restrictions on the coefficients implied by the speculative bubble model while the third panel of Table 1 presents the results of the likelihood ratio (LR) tests of the simpler volatility regimes, mixture of normals, fads and VNS models.

From the first panel of Table 1 we can see that all the coefficients of the augmented model, except the bubble coefficient of the surviving equation \( (\beta_{s,a}) \), have the correct sign and a financially meaningful magnitude. Specifically, the coefficient estimate of the mean returns in the surviving regime \( (\beta_{c,0}) \) is 1.0077, and it is highly significant implying that, when volume is normal and the bubble size is zero, the expected return in the surviving regime is 0.77% per month (9.64% on an annual basis). The corresponding value of this coefficient in the VNS model is larger (1.0085), implying a mean return in the surviving regime of 10.69% on an annual basis. On the other hand, the mean gross return in the collapsing regime \( (\beta_{c,0}) \) is -3.68% (-36.23% annualised) for the augmented model compared with -34.46% on an annual basis for the VNS model.
According to restriction (a), the periodically collapsing speculative bubble model implies that the mean returns across the two regimes must be statistically different from each other \( (\beta_{S,0} \neq \beta_{C,0}) \). From the second panel of Table 1, there are two distinct return generating regimes since the null hypothesis that the mean returns across the two regimes are the same, is rejected at the 1% level (p-value 0.0037).

Turning to the other coefficients, although the coefficient on the relative size of the bubble in the surviving regime \( (\beta_{S,b}) \) is negative\(^{15}\) and statistically insignificant (coefficient estimate -0.0035 with p-value 0.1998), it is greater than the corresponding coefficient in the collapsing regime \( (\beta_{C,b} = -0.0363) \). The speculative bubble model requires the return in the collapsing regime to be a negative function of the size of the bubble \( (\beta_{C,b} < 0) \) while the coefficient of the bubble size in the surviving regime must be greater than the corresponding coefficient in the collapsing regime \( (\beta_{S,b} > \beta_{C,b}) \). From the second panel of Table 1, the bubble coefficient in the collapsing regime is statistically smaller than zero at the 5% level (p-value of LR test 0.0353) implying that as the bubble size increases, the returns in the collapsing regime are more negative\(^{16}\). Furthermore, we can see that restriction (c) is satisfied since \( \beta_{S,b} > \beta_{C,b} \) at the 10% level (in the second panel of Table 1 the p-value of the LR test 0.0568).

If we examine the estimates of the bubble coefficients of the state equations of the VNS model we see that, again, in the surviving state the bubble coefficient is negative and statistically insignificant (-0.0028 with a p-value of 0.2523). In the collapsing regime the bubble coefficient is smaller than in the surviving regime and is approximately equal to the augmented model estimate. This implies that if abnormal volume is zero, the two models will yield approximately the same expected returns.

However, the augmented model incorporates abnormal volume in the surviving equation. The point estimate of the abnormal volume coefficient in the surviving state \( (\beta_{S,v}) \) is statistically significant (p-value 0.0001), and has the expected sign according to restriction (f). The likelihood ratio test shows that the coefficient is positive at the at the 1% level. This implies that as volume increases the expected returns

\(^{15}\) It is not possible to derive an expected sign for this coefficient and the speculative bubble model only implies that it should be greater in value than the bubble coefficient in the collapsing regime. Nevertheless, we should expect that as the bubble increases in size, investors demand a higher return to compensate them for the increased risk of bubble collapse.

\(^{16}\) The opposite holds for negative bubbles since they collapse by yielding positive abnormal returns.

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for the next period increase. This result is consistent with our conjecture that increased abnormal volume signals increased risk and thus investors demand a higher return. For example, in September 1929, the dividend multiple bubble deviation measure was equal to 23.77% and the total volume for this month was 18.30% higher than the 6-month moving average. The expected return in the surviving regime for the next time period was 0.78% for the VNS model and 0.91% for the augmented model\(^{17}\) (9.77% and 11.54% on an annual basis respectively). The expected return for the collapsing regime for the VNS model was -4.27% and -4.54% for the augmented model (-40.73% and -42.75% on an annual basis respectively). This difference in expected returns is a direct result of the inclusion of abnormal volume in the surviving state equation. The real difference of our model, however, lies the modelling of the classifying equation that gives the probability of switching regimes.

The coefficient estimates of the classifying equation (16) for the augmented model and for the VNS model are in favour of the presence of periodically collapsing speculative bubbles. As the bubble grows, the probability of being in the surviving regime in period \(t+1\) decreases since the coefficient on the absolute bubble size is negative. For the augmented model, the intercept coefficient \((\beta_{q,0})\) implies that there is a 2.31% of switching regimes that is independent of the bubble size. This probability is calculated as \(1 - \Omega(\beta_{q,0})\) using the point estimates shown in Table 1. The corresponding probability for the VNS model is 3.20%.

The point estimate of the bubble coefficient for the augmented model in the classifying equation \((\beta_{q,b})\) is -1.2653 and is highly significant. Furthermore, the LR test shows that restriction (d), which states that this coefficient should be negative \((\beta_{q,b} < 0)\), is satisfied at the 1% level as seen from part two of Table 1. The size of \(\beta_{q,b}\) is not very different between the two models, and thus the estimated probabilities of survival, if we only examine the relative size of the bubble. The point estimate of the abnormal volume coefficient in the probability equation \((\gamma_{q,V})\) is negative as expected and is statistically significant (-0.5258 with p-value 0.0063). The LR test rejects the hypothesis that abnormal volume does not affect the probability of collapse and confirms that the probability of survival is a negative function of abnormal volume. Therefore, the probability of switching should increase significantly prior to a bubble collapse if the augmented model is superior at forecasting regime changes.

\(^{17}\) Expected returns are calculated from the point estimates of the coefficients in Table 1.
Indeed in August 1987 the probability of collapse estimated from the augmented model increases by 20.35%, to a value of 7.53% which is 340% higher than if the size of the bubble was zero and volume was normal. The VNS model estimated probability of collapse, for the same month, increases by only 8.82% to a value of 7.28% which is only 227% higher than if the bubble size was equal to zero. The S&P 500 decreased by 3.54% in the next month and by 27.47% during the next four months. The probabilities of collapse from both models, for the dividend multiple measure of fundamental values, are presented in Figure 2. It is apparent that the augmented model yields a probability of collapse that is significantly more variable than the VNS model probability of collapse. Furthermore, the probability of being in the collapsing regime increases significantly before several bubble collapses, namely August 1929, June 1932, June 1982 and October 2000. This implies that the augmented model, incorporating an abnormal volume measure, is helpful in timing bubble collapses.

Finally, the standard deviations of the error terms are consistent with the theory of speculative bubbles since they should have greater variance in the collapsing regime than in the surviving regime. This is because bubbles often collapse by yielding extreme negative returns (or positive returns in the case of price decreasing bubbles). The error standard deviation in the surviving regime is 3.21% while in the collapsing regime it is 10.51% on a monthly basis.

The third panel of Table 1 presents the results of the LR tests of the augmented model against simpler models that capture well documented properties of stock market returns. The LR test for the volatility regimes alternative rejects the volatility regimes model at the 1% level implying either that the mean returns are different across the two regimes, or that the bubble deviation has predictive power for the returns of period \( t+1 \) or the probability of switching regimes. Alternatively both of the restrictions may not be supported by the data. The test for the mixture of normal distributions separates the two restrictions and the result of the LR test shows that the data reject the mixture of normals alternative in favour of the periodically collapsing speculative bubble model. This shows that the measure of bubble deviations indeed has significant forecasting ability over the returns of the next period and the probability of

---

18 This probability is calculated as \( 1 - \Omega(\beta_{y,0} + \beta_{y,1}|B_t| + \gamma_{y1}V_{y1}^*) \) for the augmented model and \( 1 - \Omega(\beta_{y,0} + \beta_{y,1}|B_t|) \) for the VNS model using the point estimates of Table 1.

19 Note that some of these periods were followed by market rallies. This is because we are also examining price decreasing bubbles, which collapse yielding positive returns. The probabilities of collapse for both models are also high in other periods that were not followed by bubble collapses. This could be evidence against the speculative models. However, we will show in the next Section that the switching bubble models have significant predictive ability and can be used to time market reversals.
switching regimes. The LR test statistic is 28.69, signifying rejection of the null of the mixture of normal distributions at the 1% significance level.

We also examine the augmented model against a model of simple mean reversion in the S&P 500 returns. Again, the fads model is strongly rejected in favour of the periodically collapsing speculative bubble. The implication of this rejection is that returns are a non-linear function of the bubble deviations, or that bubble deviations can help classify returns into two regimes, or both. Alternatively, the mean returns could be significantly different across the two regimes. Note that the above results are consistent with the results of VNS even in this larger sample that contains the large bubble deviations of the 1990’s.

As a final and more important statistical test, we examine whether abnormal volume has any explanatory power over the level of next periods returns and the generating state of returns. This is done by examining whether both abnormal volume coefficients are equal to zero. As seen from the third panel of Table 1, the data reject the hypothesis that the measure of abnormal volume has no explanatory or classifying power in our switching regression model at the 1% level. This shows that abnormal volume is significant in explaining expected returns and that it can be used to forecast the regime of the next time period.

In order to insure that our model is robust to alternative specifications of fundamentals, we re-estimate the model using the bubble deviations calculated from the Campbell and Shiller measure of fundamental values. These fundamental values allow for predictable variation in the dividend price-ratio. The results for both models under this alternative fundamental specification are presented in Table 2 and are roughly unchanged. In the first panel of Table 2 we present the results of both speculative bubble models. The constant coefficient estimates across the two regimes ($\beta_{s,0}$ and $\beta_{c,0}$) are 1.0082 and 0.9779 respectively compared to 1.0077 and 0.9632 for the dividend multiple measure of fundamental values. The LR tests presented in the second panel of Table 2 show that restriction (a) ($\beta_{s,0} \not= \beta_{c,0}$) is again supported (that is, the null hypothesis that $\beta_{s,0} = \beta_{c,0}$ is rejected) at the 5% level. The point estimate of the bubble coefficient in the collapsing equation ($\beta_{c,b}$) is smaller than zero (-0.0194) although now it is statistically insignificant (p-value 0.3670). The bubble coefficient in the surviving equation ($\beta_{s,b}$) is negative but statistically insignificant and the restriction that $\beta_{s,b}$ should be greater than $\beta_{c,b}$ is not supported by the
data (the null hypothesis that $\beta_{S,b} = \beta_{C,b}$ is not rejected)$^{20}$. The results for the VNS model also show that the statistical significance of the bubble coefficients in the state equations diminishes if we allow for time variation in the dividend growth rate.

Although under this specification of fundamentals the bubble deviations do not appear to have explanatory power over the expected returns, they still have significant power in predicting the generating state of returns. From the results of Table 2, $\beta_{q,b}$ is smaller than zero and highly significant, as shown by the result of the $t$-test and the LR test. However, $\beta_{q,b}$ is now smaller in value ($-2.1811$). The size of this coefficient implies that, as the absolute size of the bubble increases, the probability that the bubble will continue to exist is significantly smaller than under the previous measure of bubble deviations. This could be caused by the fact that bubble deviations calculated from the Campbell and Shiller fundamental values are smaller on average than the dividend multiple measure of bubble deviations.

More importantly, the effect and the significance of abnormal volume is unchanged and $\beta_{S,V}$ is positive and statistically significant. Furthermore, $\gamma_{q,V}$ is negative ($-0.5841$) and statistically smaller than zero (LR test $p$-value 0.0021). The results show that the bubble deviations from fundamental values and the deviation of volume from the six month moving average have predictive ability over the generating state and the level of next period’s returns.

Finally, the LR tests of the robustness of the speculative bubble models against stylised alternatives presented in the third panel of Table 2 show again that the augmented speculative bubble model captures effects that are not captured by the other, more parsimonious, models. More significantly, the VNS LR test rejects the hypothesis that volume does not affect the levels or the generating state of future returns at the $1\%$ level$^{21}$.

$^{20}$ Note that in the original VNS results, the bubble coefficient in the surviving equation was never significant, regardless of the specification of fundamentals and the significance of the bubble coefficient in the collapsing regime diminished under the Campbell and Shiller measure of fundamental values.

$^{21}$ However, VNS estimate their model using part of our sample (January 1926 - December 1989). In order to directly examine the validity and statistical significance of our model, we re-estimate our model and using the original VNS sample. Again, in this sub-sample the abnormal volume measure is highly significant both as a risk factor in the surviving state equation and as a classifying variable in the transition equation for both measures of fundamental values. We have also examined our model for different sub samples (namely: 1888-1926, 1888-1948, 1926-1954, 1954-1974, 1974-1989, 1974-2001, 1948-2001) in order to examine the robustness of the model and the
6 PREDICTIVE AND PROFITABILITY ANALYSIS

In the previous sections we showed that the augmented regime switching speculative bubble model has explanatory power over the S&P 500 returns. However we have said nothing about the ability of this model to forecast historical bubble collapses. In a previous study, Van Norden and Vigfusson (1998) examined the size and the power of bubble tests based on regime switching models, and found that the tests are conservative, but have significant power in detecting periodically collapsing speculative bubbles. However, their technique only examines the econometric reliability of the switching speculative behaviour model developed by VNS.

In this section we will examine the out of sample forecasting ability of the augmented model and of the VNS model and investigate whether regime switching speculative bubble models can be used to create trading rules that could yield abnormal trading profits. Although several bubble tests have been created, to our knowledge, all of them have been targeted at the identification of bubble presence and none of them have examined whether inferences from these tests can be used in order to make financially meaningful forecasts. This approach will also help examine the predictive ability of our model against that of the VNS model, in a financially intuitive way.

In order to insure that the trading rules are formed using only information that is available to investors in real time, we cut the sample approximately in half, and use the data from January 1888 to December 1947 to get initial estimates of the apparent bubble deviations and of the coefficients of the augmented and VNS models. Using the point estimates of the two models, we then calculate the conditional probability of an unusually low and of an unusually high return for the next month (January 1948). We then proceed to update our sample by one observation and re-estimate the models and the probabilities of a crash and of a rally. We continue updating the sample period used by one month until the end of the sample (January 2001) is reached.

Note that we consider both probabilities since we allow for positive and negative bubbles, and positive (negative) bubbles collapse by yielding positive (negative) returns. Using this rolling estimation, we are able to form forecasts from the two models using only information that is available to investors up to that point in time. We then proceed to form trading rules from the inferences from both models and calculate results are roughly unchanged. The results for these samples are not presented here for brevity and are available
the risk and return of the strategies at each point in time. These trading rules are based on observation of
the expected probability of a crash (rally) in the case of a positive (negative) bubble. We then evaluate
the trading rules by calculating the profits (or losses) an investor would have made if he were using the VNS
model or the augmented model in an effort to time large market movements from January 1948 to January
2001. In order to calculate the conditional probability of a crash we use the following equation:

\[
\Pr (r_{t+1} < K)_t = q(r_{t+1} \mid S)_t \omega \left( \frac{K - \beta_{s,b,t} B_t - \beta_{s,y,t} V^x_t}{\sigma^s_{u,t}} \right) + q(r_{t+1} \mid C)_t \omega \left( \frac{K - \beta_{c,b,t} - \beta_{c,Y,t} B_t}{\sigma^c_{u,t}} \right)
\]  

(32)

where \( q(r_{t+1} \mid S)_t = \Omega(\beta_{b,q,t} + \beta_{b,y,t} B_t + \gamma_{q,Y,t} V^x_t) \), \( q(r_{t+1} \mid C)_t = 1 - q(r_{t+1} \mid S)_t \), the time subscript
attached to the coefficients denote the estimated values of the coefficients using data only up to and
including time \( t \), \( K \) is the threshold below which a return is classified as a crash given
by \( K = \mu_t - 2^\ast (\sigma_{r,t}) \), \( \mu_t \) is the mean of past gross returns until time \( t \), and \( \sigma_{r,t} \) is the standard
deviation of past gross returns until time \( t \). The conditional probability of observing an extreme positive
return of at least two standard deviations above the mean of past returns, in period \( t+1 \) can be defined
similarly.

The probabilities of a crash derived from the VNS model and from the augmented model for the dividend
multiple measure of fundamental values are presented in Figure 3 together with markers to signify the 20
largest negative 1-month and 3-month returns observed for the S&P 500\(^{22}\). In the top part of the figure,
we plot the logarithm of the real S&P 500 index and the logarithm of the dividend multiple measure of
fundamental values. From Figure 3 we can see that the probability of a crash increases during several
periods when a bubble is suspected to be present but more importantly it is high before several of the 20
largest 1-month declines of the S&P 500. More specifically, we note that the probability of a crash
increased by 18.20% in August 1987 to a value of 4.41% suggesting that a bubble collapse was likely in
September 1987. The corresponding probability from the VNS model increased by only 10.09% to a
value of 4.24%. The average probability of a crash estimated using the augmented model for the previous
year was 3.11% while for the VNS model the average probability of a crash was 3.49%. Although the

\(^{22}\) At this point, it should be noted that we have allowed for partial collapses in the specification of the bubble
model and therefore a bubble may partially collapse for several periods before starting to grow again. For this
reason, we also examine the probability of a crash (rally) against the top 20 3-month negative returns as well as the
top 20 draw-downs. A draw-down is defined as the cumulative return from the last local maximum to the next local
minimum of the S&P 500 Index and thus refers to cumulative continuous losses.
market did not crash until October 1987, the market declined for four consecutive months at the end of 1987 starting in September, and thus the behaviour of the probability of a crash could be taken as evidence that the augmented model times bubble collapses more sharply than the VNS model. The probability of a crash is also high in several other periods including 1962, 1970 and 2000. All of these periods are followed by a strong correction in stock price levels.

In Figure 4 we present the conditional probability of a rally calculated from the augmented model and the VNS model alongside with markers to signify the 20 largest 1-month, 3-month and consecutive market advances. The probability of a rally increases dramatically during several periods when a negative bubble appears to be present, especially in 1949-1950 and 1982. Again, the probability of a rally estimated from the augmented model is significantly more variable than the corresponding probability derived from the VNS model. The same conclusions can be drawn by examining the probabilities of a crash and of a rally produced by the Campbell and Shiller measure of fundamental values presented in Figure 5 and Figure 6. The probability of a crash spikes prior to market corrections and the probability of a rally increases significantly before negative bubble collapses.

However, we can see that there are several periods during which both probabilities increase simultaneously thus damping the effects we seek to observe, namely the conditional probability of a crash and of a rally in the next period. This is because the conditional distribution of expected returns is a mixture of a low variance (surviving state) and a high variance (collapsing state) distribution. As the relative size of the bubble increases, the weight of the high variance distribution increases and thus both tails increase at the same time.

Based on the conditional probabilities of a crash and of a rally from both the augmented model and the VNS model, we form a market-timing rule that can be used by the investor to determine when to be in or out of the market. The trading rule states that when the probability of a crash (rally) crosses the upper 90% percentile of its historical value, the investor should sell (buy) the index, invest in the risk free

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23 The 20 largest consecutive market advances (‘draw-ups’) are defined as the 20 largest consecutive positive returns, defined as the return from the last local minimum to the next local maximum.

24 Focusing on the 90th percentile is somewhat arbitrary, but represents a trade-off between using too high a cut-off which will encourage the investor to remain in the market when the bubble has a historically high probability of collapse, while using too low a cut-off will lead the investor out of the market too frequently, resulting in missed bull market opportunities. Our results are not qualitatively altered if an 80% or 95% cut-off is employed instead.
asset (equities)\textsuperscript{25}, and maintain this position until the probability of a crash (rally) becomes lower than its historical median value, i.e. until the bubble deflates\textsuperscript{26}. When the appropriate probability becomes lower than its historical median value, the investor’s entire wealth should be placed in the S&P 500 Index. We include the probability of a rally in the strategy since an investor should buy if there is a negative bubble and the probability of a rally is greater than the 90\% percentile of its historical values. In order to insure that we are not using any information that is not available to the investor at time \( t \), we calculate the median value and the top 90\textsuperscript{th} value using a rolling window with a fixed starting point in January 1888.

We compare the two models by calculating the total holding period return for every month and examine the mean, standard deviation, skewness and kurtosis of each trading rule’s return distribution. We note the number of trades the trading rule has generated over the trading period in order to adjust trading profits for transaction costs, and we also note the percentage of time that an investor following the rule would have invested in equities. To take into account transaction costs, we assume a 0.5\% round trip cost paid upon exit from the market. We then compare the trading performance of the augmented model with the results of the VNS model and with the results of a simple buy and hold strategy.

Furthermore, in order to examine the statistical significance of the profits generated from the trading rule, we form 10,000 long random trading rules created by randomly generating series of zeros and ones, the length of which is equal to the number of months in our trading sample (January 1946–January 2001, or 661 months) using a binomial distribution. The probability of success (i.e. of a binomial draw of one) is set equal to the percentage of time that trading rule would suggest the investor to be in the market. We use this probability of success because it yields random trading rules with comparable average holding periods to our trading rules. To test for the statistical significance of the bubble rules we compare the risk adjusted returns and the other moments of the returns’ distributions with those of the random rules and if our model yields a risk adjusted profit larger than 90\%, 95\% or 99\% of the random trading rules we can conclude that our abnormal profits are statistically significant at the 10\%, 5\% and 1\% levels respectively. Finally, for every trading rule generated from the bubble models and the random rules, we calculate the wealth that an investor would have accumulated in January 2001, from an initial investment of one dollar in January 1948.

\textsuperscript{25} The risk free rate for the period January 1948 to January 2001 is taken to be the monthly continuously compounded yield on 3-Month Treasury Bills. The data are taken from the Federal Reserve Bank of St Louis web site \url{http://www.stls.frb.org/fred/data/irates.html}. 

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The results of the augmented model and the VNS model trading rules are presented in Table 3 alongside the results of the buy and hold strategy. The figures in parentheses show the percentage of randomly generated rules that would have led to higher average returns, lower standard deviation, higher skewness etc. Thus, in each case, the lower the percentage value in parentheses, the better the relative performance of the bubble trading model would have been. The second column of Table 3 contains the average of the real total monthly returns of the speculative bubble models’ trading rules and the third column the standard deviations of the total returns. Overall, we note that the augmented model trading rule achieves higher risk adjusted returns since it yields higher average returns than the VNS model with lower standard deviations. For example, using the dividend multiple measure of fundamental values, an investor would have received an average return of 0.55% per month (6.81% continuously compounded annualised return) with a standard deviation of 1.78% if he was using the augmented model compared with 0.46% (5.66% annualised) if he was using the VNS model (standard deviation 2.11%). The difference in average returns increases if the investor was using the Campbell and Shiller model to estimate fundamental values.

Furthermore, the average return of the augmented model trading rule is statistically significant, since it is greater than the mean return of 99.9% of the random trading rules, while the VNS model only manages to beat 93.66% of the randomly generated trading rules. The superiority of our model is clearer if we examine the reward to variability ratio, since the Sharpe ratio for our model is higher than the VNS model’s ratio and higher than the Sharpe ratio of 99.56% of the random trading strategies with the same percentage of time in the market. Moreover, the Sharpe ratio of the augmented model trading rule is higher than the Sharpe ratio of the buy and hold strategy, and this superiority does not fade if we examine the total wealth and the Sharpe ratio after we take into account the transaction costs involved. For both measures of fundamental values, the augmented model trading rule yields significantly higher end of period wealth than the VNS model. This higher end of period wealth is achieved with higher skewness and lower kurtosis coefficients. Both of these higher moments of the distribution of augmented model returns would be more desirable to investors than those of the VNS model under some fairly weak assumptions concerning the shape of investor utility functions (see Scott and Horvath (1980) for higher moment preferences and Kraus & Litzenberer (1976) and references therein for skewness preference in asset pricing). For example, the augmented model with the dividend multiple measure of fundamental values, yields 74% higher end of period wealth than the VNS model ($33.53 against $19.25), with a

---

26 We use the median in order to avoid any unwanted influence from extraordinarily large probabilities of a crash and of a rally observed during the sample period (especially 1929-1933).
higher Sharpe ratio (0.28 compared with 0.16), positive skewness (instead of negative from the VNS model) and lower kurtosis, with a similar number of trades at 11 (versus 10). Again, the augmented model’s end of period wealth and Sharpe ratio are statistically significant since they are higher than the end of period wealth of 99.9% and the Sharpe ratio of 99.56% the random trading rules.

The skewness of the distribution of total returns of the augmented model is significantly higher than the skewness of the buy and hold strategy, although the kurtosis coefficients of the speculative bubble model trading rules’ returns are also higher. However, the end of period wealth of the buy and hold strategy is significantly higher than the portfolio value of the augmented model trading rule ($59.71 compared with $33.53 and $37.92). This implies that if an investor used the augmented model to time entry to and exit from the market, he would have significantly less wealth by January 2001. Nevertheless, examining the augmented model trading rule in more detail, shows that this lower end of period wealth is caused by the large bubble deviation that is observed towards the end of the sample. This large and persistent bubble does not crash in our sample and causes the speculative bubble model to produce a large probability of a crash throughout the 1990’s. Figure 7 plots the real S&P 500 Composite Index and the net excess wealth of the augmented model trading rule as a percentage of the wealth of the buy and hold strategy. On the plot of the S&P 500, we place markers that signify entry and exit times that the augmented model trading rule has generated using the dividend multiple measure of fundamentals. In this figure, investor wealth has been adjusted for transaction costs assuming a 0.5% round trip cost paid upon exit from the market.

From the figure it is evident that the augmented model, estimated using the dividend multiple measure of fundamentals, forces the investor to be out of the market for long periods of time (especially in the 1960’s and 1990’s). Nevertheless, it produces higher wealth than the buy and hold strategy until October 1995. At the end of the sample, however, the buy and hold strategy yields significantly higher wealth since the large observed bubble deviation does reverse during the sample period employed. Furthermore, the augmented model underperforms the buy and hold strategy in the 1960’s, since the fundamental values produce a large and persistent bubble that collapses in 1970. This causes the augmented model to produce large probabilities of extreme returns and thus the trading rule produces a sell signal for long periods of time, leading the investor to be out of the market 75% of the time.

Examining the trading rule results of the augmented model using the Campbell and Shiller fundamental values, we note that the augmented model trading rule produces a persistent sell signal for shorter periods of time (see Figure 8). Again, the augmented model trading rule yields a higher wealth than the buy and hold strategy from September 1974 to September 1996. The augmented bubble trading rule is unable to
generate higher returns than a buy and hold strategy post-1996, and there are a number of possible reasons for this. First, it is possible that a bubble of the form assumed was not present in the data at the end of the sample period or it is possible that the model used to estimate fundamental values is not adequate and does not capture fundamentals precisely. An alternative explanation is that this result is the consequence of a manifestation of the “peso problem”, where the speculative bubble models suggest that a market crash is imminent, but do not suggest a precise date when this will occur. Such an event did not occur during the sample period, but this does not mean that it was wrong to predict that it would; indeed, the market was subject to substantial falls over much of the subsequent 1-year period after the end of the sample.

7 CONCLUSIONS

In this study we have presented a switching regime speculative bubble model that is able to capture speculative dynamics present in the S&P 500 Composite Index. The model draws from collapsing bubble theory and from the model developed by Van Norden and Schaller (1999), and augments the VNS model by including a proxy for market beliefs concerning the probability of the bubble continuing to exist. This proxy is the observed abnormal volume in the last time period that is assumed to be a signal to investors that other market participants are selling the bubbly asset. This implies that abnormal volume could be used to time bubble collapses.

The results of this model are encouraging. Using data on the S&P 500 for the period January 1888 – January 2001, we find that the bubble deviation of actual prices from fundamental values and abnormal volume, measured as the percentage deviation of volume from the 6-month moving average, has significant explanatory power for gross stock market future returns, and can help classify gross returns into a bubble surviving and a bubble collapsing regime. More specifically, as the bubble grows in size, the probability of being in a bubble collapsing regime in the next time period increases and this probability is also a positive function of abnormal volume. We also find that abnormal volume indirectly affects future expected returns as a risk factor, a result that is consistent with previous research. We find that the probability of observing an extreme negative return, of at least two standard deviations below the historical mean of returns, increases significantly when a positive bubble is present and abnormal volume is high.

We also examine the robustness of our model against different specifications of fundamental values and find that our model is robust to fundamental values that allow for predictable variation in the dividend.
growth rate and thus predictable variation in the fundamental price-dividend ratio. When the augmented model is tested against more parsimonious and established alternatives such as fads, mixtures of normal distributions, volatility regimes and the original model of Schaller and Van Norden, it appears that our model’s specification captures additional information present in the data and classifies returns in the switching regime framework more efficiently.

Although Van Norden and Vigfusson (1998) examine the statistical power and reliability of regime switching bubble models, we test the out-of-sample forecasting ability of the augmented model and the VNS model in a financially intuitive way. We construct trading rules based on inferences about the conditional probability of a crash and of a rally, and analyse the risk-adjusted returns obtained with the use of the VNS model and the augmented model. In order to ensure that we have not used any data that is not available to investors, we estimate the VNS and augmented model using rolling regressions with a fixed starting point. We examine the timing ability of the bubble models by comparing the returns of the speculative bubble model trading rules with the returns on 10,000 randomly generated trading rules that have the same average proportion of the sample period invested in equities. We find that the augmented model can consistently lead to higher risk-adjusted returns than the VNS model and the randomly generated trading rules, although it does not beat the buy and hold strategy in risk-unadjusted (total end of period wealth) terms.

This result is mainly attributed to the fact that the estimated fundamental values yield a significant and persistent positive bubble deviation in the 1990’s, thus causing the augmented model to produce large conditional probabilities of a crash. Therefore, the augmented model trading rule forced the investor to remain out of the market for protracted periods of time, while a correction did not occur by the end of our sample period. An alternative approach would be to allow the observed bubble deviations to follow a different generating process than the one described by Blanchard and Watson (1982) and used by VNS. More specifically, it could be the case that bubble deviations are systematic and persistent during long periods of time before entering a ‘critical’ state in which the probability of collapse is significantly higher. A bubble process that could produce such ‘distinct’ periods of deterministic and explosive growth has been described in Evans (1991), and the investigation of the empirical usefulness of such speculative bubble models appears to be a fruitful avenue for further research.
8 REFERENCES


9 TABLES AND FIGURES


\[
\begin{align*}
    r_{t+1} &= \beta_{b,0} S_t + \beta_{b,b} B_t + \beta_{b,v} V_t^v + u_{t+1}^S, \\
    r_{t+1} &= \beta_{c,0} C_t + \beta_{c,b} B_t + u_{t+1}^C, \\
    q(B_t, V_t^v) &= \Omega(\beta_{q,0} + \beta_{q,b} B_t + \gamma_{q,v} V_t^v).
\end{align*}
\]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Augmented Model</th>
<th>Van Norden and Schaller Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{b,0})</td>
<td>1.0077</td>
<td>1.0085</td>
</tr>
<tr>
<td>(\beta_{b,b})</td>
<td>-0.0035</td>
<td>-0.0028</td>
</tr>
<tr>
<td>(\beta_{b,v})</td>
<td>0.0123</td>
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</tr>
<tr>
<td>(\beta_{c,0})</td>
<td>0.9632</td>
<td>0.9654</td>
</tr>
<tr>
<td>(\beta_{c,b})</td>
<td>-0.0363</td>
<td>-0.0339</td>
</tr>
<tr>
<td>(\beta_{q,0})</td>
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<td>1.8519</td>
</tr>
<tr>
<td>(\beta_{q,b})</td>
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<td>-1.1793</td>
</tr>
<tr>
<td>(\gamma_{q,v})</td>
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<tr>
<td>(\sigma^S)</td>
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</tr>
<tr>
<td>(\sigma^C)</td>
<td>0.1051</td>
<td>0.1018</td>
</tr>
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Speculative Bubble Model Restrictions LR Tests

<table>
<thead>
<tr>
<th>Restriction</th>
<th>LR Test Statistic</th>
<th>p-value</th>
<th>LR Test Statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{b,0} \neq \beta_{c,0})</td>
<td>8.4110</td>
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<td>9.2120</td>
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<td>2.5048</td>
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<td>0.0560</td>
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<tr>
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<td>0.0000</td>
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<tr>
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<td>7.4111</td>
<td>0.0032</td>
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<tr>
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Robustness of Speculative Bubble Models Against Stylised Alternatives LR Tests

<table>
<thead>
<tr>
<th>Volatility Regimes</th>
<th>LR Test Statistic</th>
<th>p-value</th>
<th>LR Test Statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixture of Normals</td>
<td>47.8535</td>
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<tr>
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<td>28.6987</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0000</td>
<td>30.7171</td>
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</tr>
<tr>
<td></td>
<td>21.1731</td>
<td>0.0000</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The Van Norden and Schaller model coefficients are estimated using the model described in equation (13) in the text. The p-values are calculated using standard errors estimated from the inverse of the Hessian matrix at the optimum. The volatility regimes test imposes the restrictions: \(\beta_{b,0} = \beta_{c,0}\) and \(\beta_{b,b} = \beta_{c,b} = \beta_{q,b} = 0\). The mixture of normals test imposes the restrictions \(\beta_{b,0} = \beta_{c,0}\) and \(\beta_{b,b} = \beta_{c,b} = \beta_{q,b} = 0\). The fads test imposes the restrictions \(\beta_{b,0} = \beta_{c,0}\) and \(\beta_{b,b} = \beta_{c,b}\) and \(\beta_{q,b} = 0\). The VNS test imposes the restrictions \(\beta_{b,v} = \gamma_{q,v} = 0\).
Table 2:

\[
r_{t+1} = \beta_{s,0} + \beta_{s,b} B_t + \beta_{s,V} V_t^b + u_{t+1}^{sV}
\]
with probability \(q(B_t, V_t^b)\)

\[
r_{t+1} = \beta_{c,0} + \beta_{c,b} B_t + u_{t+1}^{c}
\]
with probability \(1 - q(B_t, V_t^b)\)

\[
q(B_t, V_t^b) = \Omega(\beta_{q,0} + \beta_{q,b} B_t + \gamma_{q,V} V_t^b)
\]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Augmented Model</th>
<th>Van Norden and Schaller Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>p-value</td>
</tr>
<tr>
<td>(\beta_{s,0})</td>
<td>1.0082</td>
<td>0.0000</td>
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<tr>
<td>(\beta_{s,b})</td>
<td>-0.0031</td>
<td>0.3349</td>
</tr>
<tr>
<td>(\beta_{s,V})</td>
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<td>(\beta_{c,0})</td>
<td>0.9779</td>
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<td>(\beta_{c,b})</td>
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<td>(\beta_{q,0})</td>
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<tr>
<td>(\beta_{q,b})</td>
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<td>(\gamma_{q,V})</td>
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<td>(\sigma_{s}^s)</td>
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<tr>
<td>(\sigma_{c}^c)</td>
<td>0.1080</td>
<td>0.0000</td>
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Speculative Bubble Model Restrictions LR Tests

<table>
<thead>
<tr>
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<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{s,0} \neq \beta_{c,0})</td>
<td>5.4374</td>
<td>0.0197</td>
<td>6.0303</td>
<td>0.0141</td>
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<td>(\beta_{c,b} &lt; 0)</td>
<td>0.1708</td>
<td>0.3397</td>
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<td>0.3088</td>
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<tr>
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<td>0.0979</td>
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<td>0.3131</td>
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<td>0.0000</td>
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<td>8.2186</td>
<td>0.0021</td>
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<tr>
<td>(\beta_{s,V} &gt; 0)</td>
<td>16.9914</td>
<td>0.0000</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Robustness of Speculative Bubble Models Against Stylised Alternatives LR Tests

<table>
<thead>
<tr>
<th>Volatility Regimes</th>
<th>LR Test Statistic</th>
<th>p-value</th>
<th>LR Test Statistic</th>
<th>p-value</th>
</tr>
</thead>
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<td>Robustness</td>
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<tr>
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</tr>
<tr>
<td>Fads</td>
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<td>22.7195</td>
<td>0.0000</td>
</tr>
<tr>
<td>VNS</td>
<td>22.5373</td>
<td>0.0000</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The Van Norden and Schaller model coefficients are estimated using the model described in equation (13) in the text. The p-values are calculated using standard errors estimated from the inverse of the Hessian matrix at the optimum. The volatility regimes test imposes the restrictions: \(\beta_{s,0} = \beta_{c,0}\) and \(\beta_{s,b} = \beta_{c,b} = 0\). The mixture of normals test imposes the restrictions \(\beta_{s,b} = \beta_{c,b} = \beta_{q,b} = 0\). The fads test imposes the restrictions \(\beta_{s,V} = \beta_{c,V} = \beta_{s,b} = \beta_{c,b}\) and \(\beta_{q,b} = 0\). The VNS test imposes the restrictions \(\beta_{s,V} = \gamma_{q,b} = 0\).

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean Return</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Excess Kurtosis</th>
<th>End of Period Wealth</th>
<th>Sharpe Ratio</th>
<th>% of Time in the Market</th>
<th>Number of (round trip) Trades</th>
<th>Adjusted Sharpe Ratio</th>
<th>Adjusted End of Period Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy and Hold</td>
<td>0.68%</td>
<td>3.48%</td>
<td>-0.54</td>
<td>1.76</td>
<td>$59.71</td>
<td>0.1815</td>
<td>100.00%</td>
<td>1</td>
<td>0.1812</td>
<td>$59.41</td>
</tr>
<tr>
<td>Risk Free Investment</td>
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<td>0.44%</td>
<td>-1.64</td>
<td>6.67</td>
<td>$1.42</td>
<td>-</td>
<td>0.00%</td>
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</table>

**Dividend Multiple Measure of Fundamentals**

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean Return</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Excess Kurtosis</th>
<th>End of Period Wealth</th>
<th>Sharpe Ratio</th>
<th>% of Time in the Market</th>
<th>Number of (round trip) Trades</th>
<th>Adjusted Sharpe Ratio</th>
<th>Adjusted End of Period Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Augmented Model</td>
<td>0.55%</td>
<td>1.78%</td>
<td>1.77</td>
<td>9.67</td>
<td>$33.53</td>
<td>0.2809</td>
<td>24.85%</td>
<td>11</td>
<td>0.2764</td>
<td>$31.73</td>
</tr>
<tr>
<td>VNS Model</td>
<td>0.46%</td>
<td>2.13%</td>
<td>-0.52</td>
<td>11.17</td>
<td>$19.25</td>
<td>0.1924</td>
<td>35.15%</td>
<td>10</td>
<td>0.1601</td>
<td>$16.77</td>
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</table>

**Campbell and Shiller Measure of Fundamentals**

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean Return</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Excess Kurtosis</th>
<th>End of Period Wealth</th>
<th>Sharpe Ratio</th>
<th>% of Time in the Market</th>
<th>Number of (round trip) Trades</th>
<th>Adjusted Sharpe Ratio</th>
<th>Adjusted End of Period Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Augmented Model</td>
<td>0.58%</td>
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<td>21</td>
<td>0.2110</td>
<td>$34.12</td>
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<tr>
<td>VNS Model</td>
<td>0.41%</td>
<td>2.58%</td>
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<td>5.34</td>
<td>$12.11</td>
<td>0.1395</td>
<td>55.45%</td>
<td>19</td>
<td>0.1189</td>
<td>$11.17</td>
</tr>
</tbody>
</table>

Trading rules are formed based on the conditional probability of a crash when a positive bubble is present and the conditional probability of a rally when a negative bubble is present. The investor either places his entire wealth in the S&P 500 Composite Index or in the 3-month U.S. Treasury Bill. The end of period wealth is the real value of the investor’s portfolio in January 2001 if the initial value of the portfolio in January 1946 was $1.00. All the numbers and the returns are in real terms. Figures in parentheses show the percentile ranking of the trading rule relative to 10,000 random trading rules with an equal percentage of time invested in the index. The mean return is the average monthly real total return and the standard deviation of returns is the standard deviation of total returns. The Sharpe ratio is the ratio of the mean excess return of a given trading rule over the corresponding standard deviation of returns. The percentage of time in the market is the percentage of months the trading rule produced a hold signal out of the 661 months in the sample. The number of trades is the total number of buy and sell orders produced by a given trading rule. The adjusted end of period wealth shows the end of period wealth net of transaction costs. Transaction costs are assumed to be 0.5% per round trip on the total value of the trade.
Figure 1:
Bubble Deviation of Actual Price from Fundamental Values.

Figure 2:
Probability of Switching Regimes in \( t+1 \): Dividend Multiple Measure of Bubble Deviations.

The probability of collapse is given by:

\[
1 - q(r_{t+1} | S) = 1 - \Omega (\beta_{q,0} + \beta_{q,1} B_t + \gamma_{q,V} V_t)
\]
Figure 3:
Probabilities of a Crash from the VNS and the Augmented Model. Dividend multiple Measure of Fundamental Values
January 1946 – January 2001

Figure 4:
Probabilities of a Rally from the VNS and the Augmented Model. Dividend multiple Measure of Fundamental Values
January 1946 – January 2001
Figure 5:
Probabilities of a Crash from the VNS and the Augmented Model.
Campbell and Shiller Measure of Fundamental Values
January 1946 – January 2001

Figure 6:
Probabilities of a Rally from the VNS and the Augmented Model.
Campbell and Shiller Measure of Fundamental Values
January 1946 – January 2001
Figure 7:

Figure 8: