A Confidence-Based Decision Rule and Ambiguity Attitudes

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We propose a decision rule — a procedure that maps incomplete judgements of an agent into final choices — that allows us to link confidence in decision-making under uncertainty to the ambiguity attitude displayed in the choice behavior. If this decision rule is applied to an affine graded preference relation (Minardi and Savochkin, 2013), the emerging choice behavior exhibits sensitivity to ambiguity and it is consistent with the generalized Hurwicz α-pessimism model studied by Ghirardato, Maccheroni, and Marinacci (2004); its famous special case of maxmin preferences of Gilboa and Schmeidler (1989) is obtained by imposing certain additional assumptions. We provide two comparative statics results: First, if the level of tolerance for the lack of confidence in comparisons decreases, the agent becomes more ambiguity averse. Second, a more decisive decision maker displays less ambiguity aversion.

Keywords: confidence, decisiveness, graded preferences, ambiguity attitude, incomplete preferences, Knightian uncertainty.

1 Introduction

Since Ellsberg's (1961) seminal thought experiment, the concept of ambiguity has been indispensable in the theory of decision making under uncertainty.[1] In this work, we explore the connection between the perception of

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[1]Following the literature, we use the term ambiguity to refer to situations in which the decision maker is unable to formulate a unique prior, and, therefore, her choices cannot be described by the standard expected utility preferences.

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ambiguity by the decision maker and her attitude towards it, on one side, and the confidence in her preferences, on the other.

As a starting point, we argue that the choice behavior violating the Subjective Expected Utility hypothesis arises when the decision maker faces difficulties in comparing uncertain prospects, and, therefore, may not easily conclude which one is superior. In this case, her intrinsic judgements are best captured by an incomplete preference relation — or, as we elaborate below, by an incomplete *graded* preference relation.[2] Then, we show that ambiguity sensitive choice behavior can arise from the use of a particular procedure — decision rule — that translates her incomplete judgements (and her confidence in them) into choices.

More formally, let $\mathcal{F}$ be the set of prospects that the decision maker evaluates and suppose that it consists of mappings ("acts") from a set Ω of the states of the world into a set X of consequences. We model the agent's decision process with the help of two primitives:

1. A graded preference relation $\mu : \mathcal{F} \times \mathcal{F} \to [0, 1]$ that captures the agent's introspection — her possibly incomplete judgments about the prospects. More precisely, $\mu(f, g)$ represents the degree of confidence that an act f is preferred over another act g.
2. A confidence-based decision rule that translates the values assigned by μ to various pairs of acts into choice behavior.

These primitives allow us to distinguish the clarity of the decision maker's mental picture of the uncertain world, and her reaction to the (lack of) clarity at the time when she needs to make choices.

In this paper, we focus on the second of the two primitives. We propose a decision rule (or, more precisely, a parametric family of decision rules) according to which the agent's choice among uncertain prospects is deter-

[2] In general, we say that agent's preferences are incomplete if she is not always sure how to rank two alternatives.

mined by where she can place them on the scale constituted by certain payoffs, how confident she is in this placement, and the parameter of the decision rule reflecting her tolerance for the lack of clarity in rankings.

As we show, if this decision rule is applied to the results of introspection that satisfy the axioms of affine graded preferences, then the final choice behavior exhibits sensitivity to ambiguity and is consistent with the generalized Hurwicz α-pessimism model studied by Ghirardato et al. (2004). As a special case, we also recover the maxmin model of Gilboa and Schmeidler (1989).

Our second set of results concerns comparative statics and allows us to pin down the relationship between confidence and ambiguity attitudes. Proposition 4 shows that if an agent has affine graded preferences, then the higher the level of confidence required to make a choice, the more ambiguity averse (in the sense of Ghirardato and Marinacci, 2002) the agent will be in his choice behavior. More importantly, Proposition 5 shows that a more decisive decision maker displays less aversion to ambiguity.

The rest of the paper is organized as follows. Section 2 introduces the formal setup and recalls the model of affine graded preferences studied in Minardi and Savochkin (2013). Section 3 defines our ambiguity sensitive decision rule and shows how it relates to the class of invariant biseparable preferences of Ghirardato et al. (2004) and maxmin preferences of Gilboa and Schmeidler (1989). Section 4 presents the comparative statics results. Section 5 concludes with an overview of the related literature. All proofs are contained in the Appendix.

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2 Affine Graded Preferences

2.1 Setup

Let Ω be a set of states of the world, endowed with an algebra Σ of events, and X be a set of consequences. We assume that X is a convex subset of a metric space, for example, the set of all lotteries over an underlying set of prizes. We denote by $\Delta(\Omega)$ the set of all finitely additive probabilities on (Ω, Σ) endowed with the weak*-topology.[3]

An act is a Σ-measurable function $f : \Omega \to X$ that takes finitely many values. We denote by $\mathcal{F}$ the set of all acts endowed with the sup-norm. With the usual abuse of notation, $x \in \mathcal{F}$ is a constant act that yields $x \in X$ for every $\omega \in \Omega$. Mixtures of acts are defined pointwise: for every $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ yields $\alpha f(\omega) + (1-\alpha)g(\omega) \in X$ for every $\omega \in \Omega$.

We occasionally use the notation $B_0(\Omega, \Sigma, \mathbb{R})$ to denote the set of all Σ-measurable functions from Ω to $\mathbb{R}$ taking finitely many values. Observe that, for any given utility index $u : X \to \mathbb{R}$, $u \circ f \in B_0(\Omega, \Sigma, \mathbb{R})$.

We model the decision maker's preferences by a *graded* (or, equivalently, *fuzzy*) preference relation $\mu : \mathcal{F} \times \mathcal{F} \to [0,1]$. Given two acts f and g, we interpret $\mu(f,g) \in [0,1]$ as the decision maker's degree of confidence to which f is at least as good as g. The extreme points of the scale correspond to situations in which the decision maker is perfectly decisive. More precisely, if $\mu(f,g) = 1$, then she is sure that f is at least as good as g; whereas, if $\mu(f,g) = 0$, she is sure that f is strictly worse than g. If the decision maker is indifferent between f and g, then $\mu(f,g) = 1$ and $\mu(g,f) = 1$. Finally, if she is indecisive, $\mu(f,g) \in (0,1)$ reflects how confident she is that f is better than g.

[3] A net $\{p_\alpha\}_{\alpha \in D}$ converges to p in the weak*-topology if and only if $p_\alpha(S) \to p(S)$ for all $S \in \Sigma$.

2.2 Axioms of affine graded preferences

In this subsection, we briefly recall the behavioral assumptions imposed on μ by the affine graded preferences model.

Reflexivity: For any $f \in \mathcal{F}$, $\mu(f,f) = 1$.

Crisp Transitivity: For any $f,g,h \in \mathcal{F}$, if $\mu(f,g) = 1$ and $\mu(g,h) = 1$, then $\mu(f,h) = 1$.

Monotonicity: For any $f,g \in \mathcal{F}$, if $\mu(f(\omega),g(\omega)) = 1$ for all $\omega \in \Omega$, then $\mu(f,g) = 1$.

C-Completeness: For any $x,y \in X$, $\mu(x,y) = 1$ or $\mu(y,x) = 1$.

Independence: For any $f,g,h \in \mathcal{F}$ and $\alpha \in (0,1]$, $\mu(f,g) = \mu(\alpha f + (1-\alpha)h, \alpha g + (1-\alpha)h)$.

Simple Dominance: For any $f,g,h \in \mathcal{F}$, if $g \succsim h$ implies $f \succsim h$ for all subjective expected utility preferences $\succsim$ that are compatible with μ, then $\mu(f,h) \geq \mu(g,h)$.[4]

Reciprocity: For any $f,g \in \mathcal{F}$, if $\mu(f,g) \in [0,1)$ then $\mu(f,g) = 1 - \mu(g,f)$.

Continuity: For all $f,g,h \in \mathcal{F}$, the mappings $\alpha \mapsto \mu(\alpha f + (1-\alpha)g, h)$ and $\alpha \mapsto \mu(h, \alpha f + (1-\alpha)g)$ are upper semicontinuous on $[0,1]$.

Nondegeneracy: There exist $f,g \in \mathcal{F}$ such that $\mu(f,g) = 0$.

The detailed discussion of these axioms can be found in Minardi and Savochkin (2013). We will refer to graded preferences that satisfy these nine axioms as *affine graded*.

2.3 Representation of affine graded preferences

Before presenting the representation, we need some technical definitions.

[4] A (crisp) binary relation $\succsim$ on $\mathcal{F}$ is said to be *compatible* with a graded preference μ if $\mu(f,g) = 1$ implies $f \succsim g$ for any $f,g \in \mathcal{F}$.

We use the term *capacity measure* for a monotone and normalized set function that is not necessarily additive: If $\mathfrak{S}$ is a collection of subsets of some arbitrary set Y such that $\varnothing \in \mathfrak{S}$ and $Y \in \mathfrak{S}$, then a function $\pi : \mathfrak{S} \to [0,1]$ is a capacity if $\pi(\varnothing) = 0$, $\pi(Y) = 1$, and $\pi(S) \geq \pi(S')$ for all $S, S' \in \mathfrak{S}$ such that $S \supseteq S'$.

For any given set $\mathcal{M}$ of finitely additive probabilities in $\Delta(\Omega)$, denote by $\mathscr{B}(\mathcal{M})$ the Borel algebra of the space $\mathcal{M}$ endowed with the relative topology.

Definition 1. Let $\mathcal{M}$ be a convex subset of $\Delta(\Omega)$. We say that $\mathcal{M}$ is *halfspace-closed* if there exist a set A and a collection $\{\varphi_\alpha\}_{\alpha \in A} \subseteq B_0(\Omega, \Sigma, \mathbb{R})$ such that $\mathcal{M} = \bigcap_{\alpha \in A} \{p \in \Delta(\Omega) : \int_\Omega \varphi_\alpha \, dp \geq 0\}$.

Definition 2. Let $\mathcal{M}$ be a nonempty subset of $\Delta(\Omega)$, and $\mathcal{L}$ be the set of affine functionals $\mathcal{M} \to \mathbb{R}$ defined as $\mathcal{L} := \{p \mapsto \int_\Omega \varphi \, dp \mid \varphi \in B_0(\Omega, \Sigma, \mathbb{R})\}$. We say that a capacity measure π on $\mathscr{B}(\mathcal{M})$ is *linearly continuous* if the mapping $\alpha \mapsto \pi(L^{-1}([\alpha, \infty)))$ is continuous on $\mathbb{R}$ for all nonconstant $L \in \mathcal{L}$ measurable with respect to $\mathscr{B}(\mathcal{M})$.

Definition 3. We say that a capacity measure π on $\mathscr{B}(\mathcal{M})$, where $\mathcal{M}$ is a nonempty, closed, and convex subset of $\Delta(\Omega)$, has *linear full support* if $\pi(\{p \in \mathcal{M} : \int_\Omega \varphi \, dp \geq 0\}) < 1$ for any $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$ such that $\int_\Omega \varphi \, dp < 0$ for at least one $p \in \mathcal{M}$.

We are ready to state the result (Minardi and Savochkin, 2013, Theorem 1) that establishes the equivalence of the axioms and the representation.

Theorem. *Let μ be a graded preference relation on $\mathcal{F}$. Then, μ satisfies Reflexivity, Crisp Transitivity, Monotonicity, C-Completeness, Independence,*

Simple Dominance, Reciprocity, Continuity, and Nondegeneracy if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$, a nonempty, convex, and halfspace-closed set $\mathcal{M}$ of probabilities in $\Delta(\Omega)$, and a linearly continuous capacity measure $\pi : \mathscr{B}(\mathcal{M}) \to [0,1]$ with linear full support such that

(i) representation

$$\mu(f,g) = \pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\right\}\right)$$

holds for all $f, g \in \mathcal{F}$;

(ii) $\pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\right\}\right) = 1 - \pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ g)\, dp \geq \int_\Omega (u \circ f)\, dp\right\}\right)$ for all $f, g \in \mathcal{F}$ such that $\int_\Omega (u \circ f)\, dp \neq \int_\Omega (u \circ g)\, dp$ for some $p \in \mathcal{M}$.

Now, we can turn attention to the choice behavior that may arise from affine graded preferences.

3 The Decision Rule

3.1 Definition of the ambiguity sensitive decision rule

This subsection introduces our decision rule — a particular procedure that takes as an input a graded preference relation and produces a binary relation that is *complete*, i.e., capable of ranking any two alternatives.

Intuitively, the decision rule that we propose is based on three postulates. First, we assume that the decision maker's introspective preferences that are captured by a graded preference relation μ already induce a complete ranking of all *certain* prospects (i.e., constant acts). Then, these certain prospects form a scale, from worse to better, that the agent can use to evaluate other prospects. (In fact, any complete, monotone, and continuous binary relation, as we formally elaborate below, is *uniquely* determined by

the placement of uncertain prospects on this scale.) Second, we postulate that the decision maker's evaluation of an act f depends on her *confidence* in comparing f with various constant acts and, consequently, the placement of f on the scale of certain outcomes; at the same time, it does not depend directly on how f is compared with other uncertain prospects. Third, the ultimate ranking of f depends on the agent's *tolerance* for the lack of clarity in her judgements and inability to make clear-cut comparisons, which is determined by $1-\gamma$, where $\gamma \in (0,1]$ is a parameter. The latter identifies the minimal level of confidence that is required to choose an alternative over a constant act.

Formally, let $\breve{\succsim}$ denote a (crisp) binary relation on X defined as

$$x \breve{\succsim} y \quad \Leftrightarrow \quad \mu(x,y) = 1 \quad \text{for all } x, y \in X.$$

Note that if μ satisfies C-Completeness, then $\breve{\succsim}$ is a complete binary relation.

Definition 4. Let γ be a real number in $(0,1]$. For any $f \in \mathcal{F}$, we denote by x_f^γ an element in X such that $x_f^\gamma \in \max_{\breve{\succsim}}\{x \in X : \mu(f,x) \geq \gamma\}$.

In words, given any act f, we consider the set of all constant acts x for which the agent is γ-sure that f is at least as good as x. Then, x_f^γ denotes a maximal element in this set.[5]

Now, we are ready to introduce a decision rule, according to which the decision maker treats x_f^γ as a "certainty equivalent" of an act f, and ranks all possible acts accordingly. For a fixed $\gamma \in (0,1]$, we define a complete preference relation $\succsim^\gamma$ on $\mathcal{F}$ as follows: For any $f, g \in \mathcal{F}$,

$$f \succsim^\gamma g \quad \Leftrightarrow \quad x_f^\gamma \breve{\succsim} x_g^\gamma. \tag{1}$$

As can be noted, this definition implies that $f \sim^\gamma x_f^\gamma$ indeed holds for all $f \in \mathcal{F}$. Moreover, it is easy to see that different values of γ correspond to

[5]The questions of the existence of a maximal element and its (lack of) uniqueness are addressed in Lemma 1 below.

different certainty equivalents x_f^γ. Indeed, the parameter γ can be thought of as the agent's confidence threshold required to make a choice; the higher the value of γ is, the more confidence is needed for the agent to move away from constant acts, and the less is her valuation of nonconstant ones.

Before moving to our main results, we state the following lemma that proves that the preference relation described by Rule (1) is well defined.

Lemma 1. *Suppose that μ is an affine graded preference and let γ be any number in $(0,1]$. Then:*

(i) For any $f \in \mathcal{F}$, the set $\{x \in X : \mu(f,x) \geq \gamma\}$ (as in Definition 4) always contains a $\breve{\succsim}$-maximal element.

(ii) The preference relation defined by Rule (1) does not depend on the choice of a maximal element.

3.2 Properties and a Representation

While Decision Rule (1) provides an intuitively appealing criterion to make decisions when μ alone is not sufficient, it is natural to ask what are the properties satisfied by preference relations defined by it. Next result provides a sharp answer to this question.

Proposition 2. *Suppose that μ is an affine graded preference relation and let γ be any number in $(0,1]$. Then, the binary relation $\succsim^\gamma$ described by Rule (1) satisfies the following axioms:*

(a) Weak Order — $\succsim^\gamma$ is complete, reflexive, and transitive;

(b) Certainty Independence — for any $f, g \in \mathcal{F}, x \in X$, and $\alpha \in (0,1)$, $f \succsim^\gamma g$ if and only if $\alpha f + (1-\alpha)x \succsim^\gamma \alpha g + (1-\alpha)x$;

(c) Archimedean Continuity — For all $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim^\gamma h\}$ and $\{\alpha \in [0,1] : h \succsim^\gamma \alpha f + (1-\alpha)g\}$ are closed in $[0,1]$;

(d) Monotonicity — if $f, g \in \mathcal{F}$ and $f(\omega) \succsim^{\gamma} g(\omega)$ for all $\omega \in \Omega$, then $f \succsim^{\gamma} g$;
(e) Nondegeneracy — there are $f, g \in \mathcal{F}$ such that $f \succ^{\gamma} g$.

Proposition 2 shows that $\succsim^{\gamma}$ is an *invariant biseparable* preference, a prominent class of preferences studied by Ghirardato et al. (2004) that satisfies all the axioms of Anscombe and Aumann (1963) except for the Independence assumption. As is well-known, C-Independence is a weakening of Independence which was introduced by Gilboa and Schmeidler (1989) to characterize their maxmin rule. Due to C-Independence, invariant biseparable preferences are sensitive to ambiguity, but they do not exhibit a specific attitude towards it because of the lack of Uncertainty Aversion. Therefore, invariant biseparable preferences encompass both the Choquet Expected Utility rule of Schmeidler (1989) and the maxmin rule of Gilboa and Schmeidler (1989). Ghirardato et al. (2004) provide a representation of invariant biseparable preferences as a generalized Hurwicz α-pessimism rule which formally separates the perception of ambiguity from the agent's reaction to it.

We now turn to determine the condition on μ which delivers an uncertainty averse decision rule. Next definition provides such condition, which, surprisingly, has to deal with transitivity of μ.

Definition 5. A graded preference relation μ is *γ-transitive* for some $\gamma \in (0, 1]$ if, for any $f, g, h \in \mathcal{F}$ such that $\mu(f, g) \geq \gamma$ and $\mu(g, h) \geq \gamma$, we have $\mu(f, h) \geq \gamma$.

In words, this definition captures the idea that if the decision maker is sufficiently confident that an option f is at least as good as g, and also sufficiently confident that g is at least as good as h, she should be sufficiently confident that f is at least good as h.[6] As a postulate, this property has

[6]There exist many other versions of transitivity in the literature on fuzzy preference

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certain appeal in many environments. At the same time, it is not universal: For instance, in certain settings, one may prefer to assume that the decision maker's confidence has a probabilistic nature, and that her confidence in the superiority of f over h should be bounded from below by the product of her confidences in the superiority of f over g and g over h.[7]

Next Proposition shows that imposing γ-transitivity on an affine graded μ is equivalent to say that $\succsim^\gamma$ satisfies the Uncertainty Aversion axiom.

> *Uncertainty Aversion*: For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \sim^\gamma g$, then $\alpha f + (1-\alpha) g \succsim^\gamma g$.

Proposition 3. *Suppose that μ is an affine graded preference relation and let γ be any number in $(0,1]$. Then, the binary relation $\succsim^\gamma$ defined by Rule (1) satisfies the Uncertainty Aversion axiom if and only if μ satisfies γ-transitivity.*

Therefore, Proposition 3 provides the conditions, under which $\succsim^\gamma$ admits a maxmin representation (Gilboa and Schmeidler, 1989):

$$f \succsim^\gamma g \quad \Leftrightarrow \quad \min_{p \in C^\gamma} \int_\Omega (u \circ f)\, dp \geq \min_{p \in C^\gamma} \int_\Omega (u \circ g)\, dp,$$

where $u : X \to \mathbb{R}$ is an affine utility index and C^γ is a nonempty and weak*-compact set of probability distributions from $\Delta(\Omega)$. Note that γ−transitivity reduces to Crisp Transitivity when $\gamma = 1$, which implies that $\succsim^1$ is always a maxmin preference relation.

The "if" part of Proposition 3 is related to Theorem 3 (and Theorem 4) of Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) since both results provide conditions for the existence of a maxmin representation. However they differ in both the approach and the objectives. Indeed, Gilboa et al.

relations and stochastic choice functions (see, e.g., Billot, 1992 and Dasgupta and Deb, 1996). Ours corresponds to Moderate Stochastic Transitivity of Fishburn (1973).

[7] See, e.g., Property (T) in Ovchinnikov (1981).

(2010) consider two binary relations and focus on the connecting properties which deliver a completion procedure of a Bewley preference to a maxmin preference. On the contrary, we start with a (graded) preference relation and derive from it a maxmin preference relation through a particular decision rule. Moreover, our emphasis is on the role played by the level γ of tolerance: in fact, as it will be clear in the next section, the parameter γ determines the size of the set of priors in the maxmin representation of $\succsim^{\gamma}$.

The "only if" part of Proposition 3 derives the uncertainty aversion property of choices from a specific form of transitivity of decision maker's introspective judgements and, to the best of our knowledge, is novel.

4 Comparative Statics

The scope of this section is to show the comparative statics exercises that one may perform using affine graded preferences together with our decision rule. More specifically, we address the following questions:

i) What is the impact of a change in the level γ of confidence on the ambiguity attitude reflected by $\succsim^{\gamma}$?

ii) Holding fixed the parameter γ, what is the impact of a change in the level of decisiveness on the ambiguity attitude?

The next proposition provides an answer to our first question by formally building a bridge between levels of confidence and ambiguity attitudes. Specifically, the higher the level of confidence γ required to make a decision, the higher the ambiguity aversion reflected into the associated choice behavior.

Proposition 4. *Suppose that μ is an affine graded preference relation, and $0 < \gamma_2 < \gamma_1 \leq 1$. Then, $\succsim^{\gamma_1}$ is more ambiguity averse than $\succsim^{\gamma_2}$ in the sense of*

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Ghirardato and Marinacci (2002): For all $x \in X$ and $f \in \mathcal{F}$,

$$x \succsim^{\gamma_2} f \quad \Rightarrow \quad x \succsim^{\gamma_1} f, \quad \textit{and}$$
$$f \succsim^{\gamma_1} x \quad \Rightarrow \quad f \succsim^{\gamma_2} x.$$

In Proposition 4 we have fixed one decision maker by μ and asked what is the effect of a change in the confidence threshold on the relative level of ambiguity aversion. Next result, instead, takes the confidence threshold as given and explores the impact of becoming more decisive on the relative ambiguity aversion. We adopt the notion of comparative decisiveness proposed by Minardi and Savochkin, which we recall next.

Definition 6. Given two graded preference relations μ_1 and μ_2 that satisfy the Reciprocity axiom, we say that μ_1 is *more decisive than* μ_2 if, for all $f, g \in \mathcal{F}$,

$$\mu_2(f,g) \geq \mu_2(g,f) \quad \Rightarrow \quad \mu_1(f,g) \geq \mu_2(f,g). \tag{2}$$

Proposition 5. *Suppose that μ_1 and μ_2 are two affine graded preference relations such that μ_1 is more decisive that μ_2. Then, for any $\gamma \in [\frac{1}{2}, 1]$, $\succsim_2^\gamma$ is more ambiguity averse than $\succsim_1^\gamma$.*

5 Related Literature and concluding remarks

The closest paper to ours is Gilboa et al. (2010). In this paper, the decision maker's behavior is described by two binary relations, $\succsim^*$ and $\succsim^\wedge$, each capturing different aspects of the decision maker's rationality. The binary relation $\succsim^*$ is incomplete and admits a representation à la Bewley, whereas the binary relation $\succsim^\wedge$ is complete and admits a maxmin representation (Gilboa and Schmeidler, 1989). In their main result (Theorem 3),

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they provide a bridge between the two models by imposing connecting axioms (Consistency and Caution) on both binary relations which guarantee that the two representations are characterized by the same utility index and the same set of priors. It is important to note that they do not need to impose Uncertainty Aversion on $\succsim\hat{}$ since their connecting assumptions already guarantee that $\succsim\hat{}$ satisfies this property. Moreover, in their Theorem 4, they further reduce the assumptions on $\succsim\hat{}$ and, at the same time, impose a stronger version of Caution (Default to Certainty) and, yet, derive the same maxmin representation for $\succsim\hat{}$. Therefore, both results provide a novel foundation for the maxmin model from the perspective of preference formation.

Cerreia-Vioglio (2012) extends the analysis of Gilboa et al. (2010) and provides a bridge between Bewley preferences and the general class of *uncertainty averse* preferences (Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011).

Another related paper to ours is Ghirardato et al. (2004). In comparison with Gilboa et al. (2010), they take a complementary approach by starting with a complete binary relation which may be ambiguity sensitive and derive from it a binary relation dubbed as "unambiguous" preference. This binary relation is typically incomplete and admits a representation á la Bewley.

As explained in the Introduction, the present paper further develops the idea of Gilboa et al. (2010) that choice behavior that displays ambiguity aversion may arise in cases when the agent experiences difficulties in comparing alternatives, and her intrinsic ranking of alternatives is incomplete. From that perspective, our model provides a finer description of the state of mind of such a decision maker, links aversion to ambiguity to her confidence in choices, and embodies a quantitative measure of that confidence.

Appendix

Let $B_0(\Omega, \Sigma)$ denote the set of all real-valued, Σ-measurable functions taking only finitely many values (simple functions) endowed with the sup-norm. For a function $\varphi \in B_0(\Omega, \Sigma)$ and a measure $p \in \Delta(\Sigma)$, let $\langle \varphi, p \rangle$ denote $\int_\Omega \varphi \, dp$.

Proof of Lemma 1. Suppose μ has the representation $(u, \mathcal{M}, \pi)$. In that representation, $x \breve{\succsim} y \Leftrightarrow \mu(x, y) = 1 \Leftrightarrow \pi(\{p \in \mathcal{M} : u(x) \geq u(y)\}) = 1$, and the latter holds if and only if $u(x) \geq u(y)$ because $\pi(\mathcal{M}) = 1$ and $\pi(\varnothing) = 0$. We conclude that $\breve{\succsim}$ is a reflexive, transitive, and complete binary relation on X because $\geq$ is a reflexive, transitive, and complete binary relation on $\mathbb{R}$. The transitivity of $\breve{\succsim}$ proves Part (ii) of the lemma: If x_1 and x_2 are two $\breve{\succsim}$-maximal elements in the set $\{x \in X : \mu(f, x) \geq \gamma\}$ for some $f \in \mathcal{F}$, then $x_1 \breve{\succsim} x_2$ and $x_2 \breve{\succsim} x_1$ (i.e., $x_1 \breve{\sim} x_2$). Therefore, $x_1 \breve{\succsim} x_g^\gamma \Leftrightarrow x_2 \breve{\succsim} x_g^\gamma$ for any $g \in \mathcal{F}$, and, symmetrically, $x_g^\gamma \breve{\succsim} x_1 \Leftrightarrow x_g^\gamma \breve{\succsim} x_2$ for any $g \in \mathcal{F}$.

Now we turn to proving Part (i). Fix $\gamma \in (0, 1]$ and $f \in \mathcal{F}$, and let $P : \mathbb{R} \to \mathbb{R}$ be defined as $P(t) := \pi(\{p \in \mathcal{M} : \langle u \circ f, p \rangle \geq t\})$.

Consider first the case in which the mapping $p \mapsto \langle u \circ f, p \rangle$ is constant on $\mathcal{M}$, i.e., there exists $c \in \mathbb{R}$ such that $c = \langle u \circ f, p \rangle$ for all $p \in \mathcal{M}$. Then, $P(t) = 1$ for all $t \leq c$ and $P(t) = 0$ for all $t > c$ because $\pi(\mathcal{M}) = 1$ and $\pi(\varnothing) = 0$. Observe that $c \in u(X)$ because u is affine and X is convex. Therefore, any $x^* \in u^{-1}(\{c\})$ is a $\breve{\succsim}$-maximal element in the set $\{x \in X : \mu(f, x) \geq \gamma\}$.

If the mapping $p \mapsto \langle u \circ f, p \rangle$ is nonconstant on $\mathcal{M}$, then P is weakly decreasing and continuous because π is monotone and linearly continuous. Since f takes only finitely many values, we can define $t_1 := \min_{\omega \in \Omega} u(f(\omega))$ and $t_0 := \max_{\omega \in \Omega} u(f(\omega))$. We observe, first, that $P(t_1) = \pi(\mathcal{M}) = 1 \geq \gamma$. Second, we note that $t_0 \in u(X)$, and, by the reciprocal property of π, $P(t_0) = 1 - \pi(\{p \in \mathcal{M} : t_0 \geq \langle u \circ f, p \rangle\})$. Therefore, $P(t_0) = 1 - \pi(\mathcal{M}) = 0$. Let $t^* := \sup\{t \in \mathbb{R} : P(t) \geq \gamma\}$. By the monotonicity of P, we have $t_1 \leq t^* \leq t_0$, and, by the continuity of P, $P(t^*) \geq \gamma$. Finally, since $t_1 \in u(X)$,

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$t_0 \in u(X)$, and $u(X)$ is convex, we have $t^* \in u(X)$. Given that, we can choose an arbitrary $x^* \in u^{-1}(\{t^*\})$, and it is, clearly, a $\breve{\succsim}$-maximal element of $\{x \in X : \mu(f,x) \geq \gamma\}$ $\square$

In the course of proving Part (i) of Lemma 1, we effectively established the following facts that we will be using later.

Claim 6. *Suppose that μ is an affine graded preference with the representation $(u, \mathcal{M}, \pi)$, let γ be any number in $(0,1]$, and $\Upsilon^\gamma : \mathcal{F} \to \mathbb{R}$ be defined as*

$$\Upsilon^\gamma(f) := \sup\Big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \geq t\}) \geq \gamma\Big\}. \tag{3}$$

Then:

(i) If $f \in \mathcal{F}$ is such that there exists $c \in \mathbb{R}$ such that $\langle u \circ f, p\rangle = c$ for all $p \in \mathcal{M}$, then $\Upsilon^\gamma(f) = c$. In particular, if $x \in X$, then $\Upsilon^\gamma(x) = u(x)$.

(ii) For any $f \in \mathcal{F}$ such that the mapping $p \mapsto \langle u \circ f, p\rangle$ is nonconstant on $\mathcal{M}$, we have $\pi(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \geq \Upsilon^\gamma(f)\}) = \gamma$.

(iii) For any $f \in \mathcal{F}$, we have $\Upsilon^\gamma(f) \in u(X)$.

(iv) For any $f, g \in \mathcal{F}$, we have

$$f \succsim^\gamma g \;\Leftrightarrow\; x_f^\gamma \breve{\succsim} x_g^\gamma \;\Leftrightarrow\; u(x_f^\gamma) \geq u(x_g^\gamma) \;\Leftrightarrow\; \Upsilon^\gamma(f) \geq \Upsilon^\gamma(g).$$

Proof of Proposition 2. Part (a), *Weak Order*. It is established in the proof of Part (ii) of Lemma 1.

Part (b), *Certainty Independence*. Fix $f \in \mathcal{F}$, $y \in X$, and $\alpha \in (0,1)$. Since u is affine, $\{p \in \mathcal{M} : \langle u \circ (\alpha f + (1-\alpha)y), p\rangle \geq \alpha t + (1-\alpha)u(y)\}) = \{p \in \mathcal{M} : \langle u \circ f, p\rangle \geq t\})$ for any $t \in \mathbb{R}$. Then, for the sets

$$A := \Big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \geq t\}) \geq \gamma\Big\} \quad \text{and}$$
$$B := \Big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ (\alpha f + (1-\alpha)y), p\rangle \geq t\}) \geq \gamma\Big\},$$

we have $t \in A$ if and only if $\alpha t + (1-\alpha)u(y) \in B$, and, therefore, $\Upsilon^\gamma(\alpha f + (1-\alpha)y) = \alpha\Upsilon^\gamma(f) + (1-\alpha)u(y)$.

To conclude the proof, we observe that

$$\begin{aligned}
&\alpha f + (1-\alpha)y \succsim^\gamma \alpha g + (1-\alpha)y \;\Leftrightarrow \\
&\Upsilon^\gamma(\alpha f + (1-\alpha)y) \geq \Upsilon^\gamma(\alpha g + (1-\alpha)y) \;\Leftrightarrow \\
&\alpha\Upsilon^\gamma(f) + (1-\alpha)u(y) \geq \alpha\Upsilon^\gamma(g) + (1-\alpha)u(y) \;\Leftrightarrow \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Upsilon^\gamma(f) \geq \Upsilon^\gamma(g) \;\Leftrightarrow\; f \succsim^\gamma g.
\end{aligned}$$

Part (c), *Archimedean Continuity.* Fix arbitrary $f, g, h \in \mathcal{F}$, and suppose that the sequence $(\alpha_n)_{n=1}^\infty$ in $[0,1]$ converges to some α. Let $d := \alpha f + (1-\alpha)g$, $d_n := \alpha_n + (1-\alpha_n)g$ for all $n \in \mathbb{N}$, and note that $u \circ d_n \to u \circ d$ in the sup-norm because f and g take only finitely many values. Our first objective is to prove that $\Upsilon^\gamma(d_n) \to \Upsilon^\gamma(d)$ as $n \to \infty$.

For an arbitrary $\varepsilon > 0$, one can find $N \in \mathbb{N}$ such that $\|(u \circ d_n) - (u \circ d)\| < \varepsilon$ for all $n > N$. Then, $|\langle u \circ d_n, p\rangle - \langle u \circ d, p\rangle| < \varepsilon$ for all $n > N$ and $p \in \mathcal{M}$, and, therefore,

$$\{p \in \mathcal{M} : \langle u \circ d_n, p\rangle \geq t - \varepsilon\} \supseteq \{p \in \mathcal{M} : \langle u \circ d, p\rangle \geq t\} \quad \text{and} \tag{4}$$

$$\{p \in \mathcal{M} : \langle u \circ d_n, p\rangle \geq t + \varepsilon\} \subseteq \{p \in \mathcal{M} : \langle u \circ d, p\rangle \geq t\} \tag{5}$$

for all $n > N$ and $t \in \mathbb{R}$. Let

$$\begin{aligned}
A &:= \big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ d, p\rangle \geq t\}) \geq \gamma\big\}, \\
A_n &:= \big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ d_n, p\rangle \geq t\}) \geq \gamma\big\}, \\
B_n &:= \big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ d_n, p\rangle \geq t - \varepsilon\}) \geq \gamma\big\}, \\
C_n &:= \big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ d_n, p\rangle \geq t + \varepsilon\}) \geq \gamma\big\},
\end{aligned}$$

and note that $\sup B_n = \sup A_n + \varepsilon$ and $\sup C_n = \sup A_n - \varepsilon$ by Claim 6. By (4)–(5) and the monotonicity of π, we have $\sup B_n \geq \sup A$ and $\sup C_n \leq \sup A$

for all $n > N$. We conclude that $\sup A_n - \varepsilon \le \sup A \le \sup A_n + \varepsilon$ — that is, $|\sup A_n - \sup A| \le \varepsilon$ — for all $n > N$, which proves that $\Upsilon^\gamma(d_n) \to \Upsilon^\gamma(d)$ as $n \to \infty$.

Now, suppose that $\alpha_n f + (1-\alpha_n)g \succsim^\gamma h$ for all $n \in \mathbb{N}$, and observe that

$$\alpha_n f + (1-\alpha_n)g \succsim^\gamma h \;\Rightarrow\; \Upsilon^\gamma(d_n) \ge \Upsilon^\gamma(h) \;\Rightarrow$$
$$\Upsilon^\gamma(d) \ge \Upsilon^\gamma(h) \;\Rightarrow\; \alpha f + (1-\alpha)g \succsim^\gamma h,$$

which proves that the set $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim^\gamma h\}$ is closed. The closedness of the symmetric set can be proven analogously.

Part (d), *Monotonicity.* Suppose that $f, g \in \mathcal{F}$ are such that $f(\omega) \succsim^\gamma g(\omega)$ for all $\omega \in \Omega$. As argued in the proof of Lemma 1, it means that $u(f(\omega)) \ge u(g(\omega))$ for all $\omega \in \Omega$. Then,

$$\{p \in \mathcal{M} : \langle u \circ f, p\rangle \ge t\} \supseteq \{p \in \mathcal{M} : \langle u \circ g, p\rangle \ge t\}$$

for all $t \in \mathbb{R}$, and, therefore,

$$\big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \ge t\}) \ge \gamma\big\} \supseteq$$
$$\big\{t \in \mathbb{R} : \pi(\{p \in \mathcal{M} : \langle u \circ g, p\rangle \ge t\}) \ge \gamma\big\}$$

by the monotonicity of π. The set inclusion implies that $\Upsilon^\gamma(f) \ge \Upsilon^\gamma(g)$, and, in turn, $f \succsim^\gamma g$.

Part (e), *Nondegeneracy.* It follows from that fact that u is nonconstant and $x \succ^\gamma y \Leftrightarrow u(x) > u(y)$ for any $x, y \in X$. □

Lemma 7. *Suppose μ is an affine graded preference relation with the representation $(u, \mathcal{M}, \pi)$, $\gamma \in (0,1]$, and $\succsim^\gamma$ is obtained by Rule (1). Then, $\succsim^\gamma$ satisfies the Uncertainty Aversion axiom if and only if, for any $\varphi, \psi \in B_0(\Omega, \Sigma)$,*

$$\left.\begin{array}{l}\pi\big(\{p \in \mathcal{M} : \langle \varphi, p\rangle \ge 0\}\big) \ge \gamma \\ \pi\big(\{p \in \mathcal{M} : \langle \psi, p\rangle \ge 0\}\big) \ge \gamma\end{array}\right\} \quad \Rightarrow \quad \pi\big(\{p \in \mathcal{M} : \langle \varphi + \psi, p\rangle \ge 0\}\big) \ge \gamma. \quad (6)$$

Proof. If part. Suppose that Condition (6) holds for all $\varphi, \psi \in B_0(\Omega, \Sigma)$, and fix arbitrary $f, g \in \mathcal{F}$. By the result of Claim 6, we have $\pi\big(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \geq \Upsilon^\gamma(g)\}\big) \geq \gamma$ and $\pi\big(\{p \in \mathcal{M} : \langle u \circ g, p\rangle \geq \Upsilon^\gamma(g)\}\big) \geq \gamma$. Then, defining $\varphi := \langle u \circ f, p\rangle - \Upsilon^\gamma(g)$, $\psi := \langle u \circ g, p\rangle - \Upsilon^\gamma(g)$, and using (6) and the affinity of u, we obtain

$$\gamma \leq \pi\big(\{p \in \mathcal{M} : \langle (u \circ f) + (u \circ g) - 2\Upsilon^\gamma(g), p\rangle \geq 0\}\big) = \pi\Big(\Big\{p \in \mathcal{M} : \Big\langle u \circ \Big(\tfrac{1}{2}f + \tfrac{1}{2}g\Big), p\Big\rangle \geq \Upsilon^\gamma(g)\Big\}\Big).$$

Therefore, $\Upsilon^\gamma(\frac{1}{2}f + \frac{1}{2}g) \geq \Upsilon^\gamma(g)$ and, in turn, $\frac{1}{2}f + \frac{1}{2}g \succsim^\gamma g$. By the continuity of $\succsim^\gamma$, this conclusion can be extended to arbitrary mixtures of f and g: For any $\alpha \in (0,1)$, $\alpha f + (1-\alpha)g \succsim^\gamma g$.

Only if part. Suppose that $\succsim^\gamma$ satisfies the Uncertainty Aversion axiom, and fix arbitrary $\varphi, \psi \in B_0(\Omega, \Sigma)$ such that $\pi\big(\{p \in \mathcal{M} : \langle \varphi, p\rangle \geq 0\}\big) \geq \gamma$ and $\pi\big(\{p \in \mathcal{M} : \langle \psi, p\rangle \geq 0\}\big) \geq \gamma$. Since u is nonconstant and $u(X)$ is convex, we can find $y \in X$ such that $u(y) \in int\, u(X)$. Since φ and ψ are bounded, we can also find a sufficiently small $\varepsilon > 0$ such that $u(y) + \varepsilon\varphi \in u(X)^\Omega$ and $u(y) + \varepsilon\psi \in u(X)^\Omega$. Let $f, g \in \mathcal{F}$ be such that $u \circ f = u(y) + \varepsilon\varphi$ and $u \circ g = u(y) + \varepsilon\psi$. Then,

$$\mu(g, y) = \pi\big(\{p \in \mathcal{M} : \langle u \circ g, p\rangle \geq u(y)\}\big) = \pi\big(\{p \in \mathcal{M} : u(y) + \varepsilon\langle \psi, p\rangle \geq u(y)\}\big) = \pi\big(\{p \in \mathcal{M} : \langle \psi, p\rangle \geq 0\}\big) \geq \gamma.$$

By the result of Claim 6, $\Upsilon^\gamma(g) \geq u(y)$. Similarly, $\mu(f, y) = \pi\big(\{p \in \mathcal{M} : \langle \varphi, p\rangle \geq 0\}\big) \geq \gamma$, and, therefore, $\Upsilon^\gamma(f) \geq u(y)$. Assume without loss of generality that $\Upsilon^\gamma(f) \geq \Upsilon^\gamma(g)$, i.e., $f \succsim^\gamma g$. Then, by assumption, $\frac{1}{2}f + \frac{1}{2}g \succsim^\gamma$

g, and we obtain

$$\gamma \le \pi\Big(\Big\{p \in \mathcal{M} : \Big\langle u \circ \Big(\tfrac{1}{2}f + \tfrac{1}{2}g\Big), p\Big\rangle \ge \Upsilon^{\gamma}(g)\Big\}\Big) \le$$
$$\pi\Big(\Big\{p \in \mathcal{M} : \Big\langle u \circ \Big(\tfrac{1}{2}f + \tfrac{1}{2}g\Big), p\Big\rangle \ge u(y)\Big\}\Big) =$$
$$\pi\Big(\Big\{p \in \mathcal{M} : u(y) + \tfrac{1}{2}\varepsilon\langle \varphi + \psi, p\rangle \ge u(y)\Big\}\Big) =$$
$$\pi\big(\{p \in \mathcal{M} : \langle \varphi + \psi, p\rangle \ge 0\}\big).$$

□

Proof of Proposition 3. *Only if* part. Suppose that the binary relation $\succsim^{\gamma}$ satisfies the Uncertainty Aversion axiom, and fix arbitrary $f, g, h \in \mathcal{M}$ such that $\mu(f,g) \ge \gamma$ and $\mu(g,h) \ge \gamma$. These two inequalities mean that $\pi\big(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \ge \langle u \circ g, p\rangle\}\big) \ge \gamma$ and $\pi\big(\{p \in \mathcal{M} : \langle u \circ g, p\rangle \ge \langle u \circ h, p\rangle\}\big) \ge \gamma$. Let $\varphi, \psi \in B_0(\Omega, \Sigma)$ be defined as $\varphi := (u \circ f) - (u \circ g)$ and $\psi := (u \circ g) - (u \circ h)$. Then, we observe that the antecedent of (6) holds, and, by Lemma 7,

$$\gamma \le \pi\big(\{p \in \mathcal{M} : \langle \varphi + \psi, p\rangle \ge 0\}\big) =$$
$$\pi\big(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \ge \langle u \circ h, p\rangle\}\big) = \mu(f,h).$$

If part. Suppose that μ satisfies γ-transitivity. Our objective is to prove that (6) holds for all $\varphi, \psi \in B_0(\Omega, \Sigma)$. Suppose $\varphi, \psi \in B_0(\Omega, \Sigma)$ are such that $\pi\big(\{p \in \mathcal{M} : \langle \varphi, p\rangle \ge 0\}\big) \ge \gamma$ and $\pi\big(\{p \in \mathcal{M} : \langle \psi, p\rangle \ge 0\}\big) \ge \gamma$. Since u is nonconstant and $u(X)$ is convex, we can find $y \in X$ such that $u(y) \in \mathit{int}\, u(X)$. Since φ and ψ are bounded, we can also find a sufficiently small $\varepsilon > 0$ such that $u(y) + \varepsilon\varphi \in u(X)^{\Omega}$ and $u(y) + \varepsilon(\varphi + \psi) \in u(X)^{\Omega}$. Let $f, g \in \mathcal{F}$ be such that $u \circ f = u(y) + \varepsilon\varphi$ and $u \circ g = u(y) + \varepsilon(\varphi + \psi)$. Then,

$$\mu(f,y) = \pi\big(\{p \in \mathcal{M} : \langle u \circ f, p\rangle \ge u(y)\}\big) =$$
$$\pi\big(\{p \in \mathcal{M} : u(y) + \varepsilon\langle \varphi, p\rangle \ge u(y)\}\big) = \pi\big(\{p \in \mathcal{M} : \langle \varphi, p\rangle \ge 0\}\big) \ge \gamma,$$

and, similarly, $\mu(g,f) \geq \gamma$. By γ-transitivity, we obtain

$$\gamma \leq \mu(g,y) = \pi\big(\{p \in \mathcal{M} : \langle u \circ g, p\rangle \geq u(y)\}\big) =$$
$$\pi\big(\{p \in \mathcal{M} : u(y) + \varepsilon\langle \varphi + \psi, p\rangle \geq u(y)\}\big) = \pi\big(\{p \in \mathcal{M} : \langle \varphi + \psi, p\rangle \geq 0\}\big).$$

Lemma 7 now delivers the Uncertainty Aversion axiom. □

Proof of Proposition 4. We observe, first, that $x_f^{\gamma_2} \breve{\succsim} x_f^{\gamma_1}$ for any $f \in \mathcal{F}$ because $\{x \in X : \mu(f,x) \geq \gamma_2\} \supseteq \{x \in X : \mu(f,x) \geq \gamma_1\}$.

If $x_f^{\gamma_1} \breve{\succsim} x$ for some $x \in X$ and $f \in \mathcal{F}$, then $x_f^{\gamma_2} \breve{\succsim} x$ since $\breve{\succsim}$ is transitive. This proves the second part of claim, $f \succsim^{\gamma_1} x$ implies $f \succsim^{\gamma_2} x$.

If $x_f^{\gamma_1} \breve{\succ} x$, then $x_f^{\gamma_2} \breve{\succ} x$, which proves that $f \succ^{\gamma_1} x$ implies $f \succ^{\gamma_2} x$. Since relations $\succsim^{\gamma}$ are complete for all $\gamma \in (0,1]$, this proves the first part of the claim, $x \succsim^{\gamma_2} f$ implies $x \succsim^{\gamma_1} f$. □

Proof of Proposition 5. Suppose that affine graded preference relations μ_1 and μ_2 are such that μ_1 is more decisive than μ_2, and $\gamma \in [\frac{1}{2}, 1]$. As proven by Minardi and Savochkin (2013, Theorem 7), the relationship between μ_1 and μ_2 implies that $\mu_1(x,y) = 1$ holds if and only if $\mu_2(x,y) = 1$ for any $x, y \in X$, which means that μ_1 and μ_2 induce the same crisp binary relation $\breve{\succsim}$ over constant acts. For any $f \in \mathcal{F}$, let x_{1f}^{γ} and x_{2f}^{γ} denote the certainty equivalents of f computed for graded preference relations μ_1 and μ_2, respectively, in accordance to Definition 4.

Now, for any $f \in \mathcal{F}$, we observe that $\{x \in X : \mu_1(f,x) \geq \gamma\} \supseteq \{x \in X : \mu_2(f,x) \geq \gamma\}$ and, therefore, $x_{1f}^{\gamma} \breve{\succsim} x_{2f}^{\gamma}$. Given this, the fact that $\succsim_2^{\gamma}$ is more ambiguity averse than $\succsim_1^{\gamma}$ can be derived by repeating the proof of Proposition 4. □

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