Second order asymptotic bias under many instruments and error non-normality

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Abstract

We consider a linear homoskedastic instrumental variables model with many instruments. In an asymptotic framework where their number is proportional to the sample size, we derive the second order asymptotic biases of the LIML and Fuller estimators when the errors may not be gaussian. We discuss the structure of bias expressions and elaborate on certain special cases.

KEYWORDS: instrumental variables regression, many instrument asymptotics, second order asymptotic bias

DECLARATIONS OF INTEREST: none.

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1 Introduction

The problem of consistent estimation of structural parameters in a linear homoskedastic instrumental variables model under many instrument sequences (Kunitomo 1980, Morimune 1983, Bekker 1994) has been successfully solved. The leading estimator, limited information maximum likelihood (LIML, Anderson and Rubin 1949), is consistent and is asymptotically more efficient than possible consistent alternatives like jackknife instrumental variables (e.g., Angrist, Imbens and Krueger, 1999) and bias corrected two-stage least squares (e.g., Donald and Newey, 2001), while the remarkable Fuller (1977) estimator resolves the problem of non-existence of LIML moments.

The higher order asymptotic properties, in particular second order asymptotic biases, are little investigated. Kunitomo (1980) derives higher order expansions for the LIML and 2SLS estimators, and Morimune (1983) develops approximations for the LIML, Fuller and modified Fuller estimators, when the structural and reduced form errors are assumed gaussian. In this paper, we derive second order asymptotic biases of the LIML and Fuller estimators within the many instrument asymptotic framework without a requirement of error gaussianity.

We find that the structure of second order asymptotic biases under error non-normality is similar to that of first order asymptotic variance derived in Hansen, Hausman and Newey (2008). Like a variance expression, a bias expression can be split into four components, one of which would appear under the traditional asymptotics, another originates from instrument numerosity under error normality, and the other two are responsible for deviations of various third and fourth error moments from their gaussian counterparts. The Fuller estimator may tuned up to remove the leading component in the LIML bias.

Online Appendix contains proofs and derivations.

2 Setup and estimators

Consider a structural equation with possibly endogenous regressors:

$$y_i = x_i'\theta_0 + e_i,$$

or in matrix notation, $Y = X\theta_0 + e$, where $Y = (y_1, ..., y_n)'$ is $n \times 1$, $X = (x_1, ..., x_n)'$ is $n \times p$, and $e = (e_1, ..., e_n)'$ is $n \times 1$. There is additionally an $n \times \ell$ matrix of instruments $Z = (z_1, ..., z_n)'$, $p \leq \ell \leq n$. Because the column dimension of Z will grow with sample size, its elements implicitly depend on n. Let the reduced form be

$$x_i = \Pi' z_i + u_i,$$

where Π is $\ell \times p$. In matrix notation, $X = Z\Pi + U$, where $U = (u_1, ..., u_n)'$ is $n \times p$. We assume that Z has full column rank and treat it as nonrandom.

The errors (e_i, u_i) are zero mean IID across i having finite eighth moments, with covariance matrix

$$\operatorname{var}\begin{pmatrix} e_i \\ u_i \end{pmatrix} = \begin{bmatrix} \sigma^2 & \Sigma' \\ \Sigma & \Omega \end{bmatrix}.$$

Denote the third and fourth moments of the structural error by

$$v_3 = E\left[e_i^3\right], \quad v_4 = E\left[e_i^4\right].$$

Let us define

$$\tilde{u}_i = u_i - \frac{\Sigma}{\sigma^2} e_i, \tag{1}$$

a population residual in the least squares projection of u_i on e_i, hence $E\left[\tilde{u}_i e_i\right] = 0$. The population 'residual variance' is

$$\tilde{\Omega} \equiv E\left[\tilde{u}_i \tilde{u}_i'\right] = \Omega - \frac{\Sigma\Sigma'}{\sigma^2}. \tag{2}$$

We impose the following many instrument asymptotic framework.

Assumption 1 *Asymptotically, as $n \to \infty$, we have $\ell/n = \alpha + o(1/n)$ with $0 < \alpha < 1$.*

Let

$$\hat{\alpha} = \frac{\ell}{n}.$$

Denote by P the projection matrix associated with Z:

$$P = Z\left(Z'Z\right)^{-1} Z',$$

with i^{th}, j^{th} element P_{ij}. Denote by D_A a diagonal matrix containing the main diagonal of square matrix A. The limited information maximum likelihood (LIML) estimator reads

$$\hat{\theta}^{LIML} = \arg\min_{\theta} \left\{ F(\theta) \equiv \frac{(Y - X\theta)' P (Y - X\theta)}{(Y - X\theta)'(Y - X\theta)} \right\},$$

and the Fuller estimator is

$$\hat{\theta}^{FULL} = \left(X'(P - \mathring{\alpha} I_n)X\right)^{-1} X'(P - \mathring{\alpha} I_n)Y,$$

where

$$\mathring{\alpha} = \frac{\tilde{\alpha} - (1 - \tilde{\alpha})\, c/(n - \ell)}{1 - (1 - \tilde{\alpha})\, c/(n - \ell)}$$

for $c > 0$, and $\tilde{\alpha} = F(\hat{\theta}^{LIML})$. The Fuller estimator is asymptotically equivalent to LIML but corrects it for the existence of moments problem. The value $c = 1$ is usually recommended.

Next we make assumptions about data generation. By 'lim' we understand taking a limit under assumption 1.

Assumption 2 *The following limits exist: $\gamma = \lim \ell^{-1} \sum_i P_{ii}^2$, $\pi_\alpha = \lim n^{-1} \Pi' Z' D_{P - \alpha I} \iota_n$, $Q = \lim n^{-1} \Pi' Z' Z \Pi$. The matrix Q is non-singular.*

The previous literature (Hansen, Hausman and Newey, 2008) establishes that with non-gaussian errors, the LIML estimator is consistent and asymptotically normal with the variance of the form

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4,$$

where $\mathcal{V}_1$ is the conventional (few instrument) component, $\mathcal{V}_2$ is the many instrument component, and $\mathcal{V}_3$ and $\mathcal{V}_4$ arise from third and fourth error moments, respectively. Namely,

$$\begin{aligned}
\mathcal{V}_1 &= \sigma^2 Q^{-1}, \\
\mathcal{V}_2 &= \frac{\alpha}{1-\alpha}\sigma^2 Q^{-1}\tilde{\Omega}Q^{-1}, \\
\mathcal{V}_3 &= \frac{1}{1-\alpha}Q^{-1}\left(E\left[\tilde{u}_i e_i^2\right]\pi_\alpha' + \pi_\alpha E\left[\tilde{u}_i' e_i^2\right]\right)Q^{-1}, \\
\mathcal{V}_4 &= \frac{\alpha\left(\gamma-\alpha\right)}{\left(1-\alpha\right)^2}Q^{-1}E\left[\tilde{u}_i\tilde{u}_i'\left(e_i^2-\sigma^2\right)\right]Q^{-1}.
\end{aligned}$$

Note that the variance components $\mathcal{V}_3$ and $\mathcal{V}_4$ vanish when the errors are jointly gaussian and/or when the main diagonal of the projection matrix P is asymptotically homogeneous (see Anatolyev and Yaskov, 2017). As the Fuller estimator is asymptotically equivalent to LIML under many instrument asymptotics, its asymptotic variance also equals $\mathcal{V}$.

3 Asymptotic biases

The bias expression $\mathcal{B}$ below divided by n is an expected difference between the estimator and the true parameter value to the second order.

Theorem: Suppose assumptions 1–2 hold. The second order asymptotic bias of the LIML estimator has the following expression:

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4,$$

where

$$\begin{aligned}
\mathcal{B}_1 &= -Q^{-1}\Sigma, \\
\mathcal{B}_2 &= -\frac{\alpha}{1-\alpha}Q^{-1}\tilde{\Omega}Q^{-1}\Sigma, \\
\mathcal{B}_3 &= \frac{1}{1-\alpha}Q^{-1}\left[\frac{v_3\tilde{\Omega}Q^{-1}}{\sigma^2} - 2E\left[\tilde{u}_i\tilde{u}_i'e_i\right]Q^{-1} - \operatorname{tr}\left(E\left[\tilde{u}_i\tilde{u}_i'e_i\right]Q^{-1}\right)\right. \\
&\qquad \left. - \frac{E\left[\tilde{u}_i e_i^2\right]\Sigma' + \Sigma' Q^{-1}E\left[\tilde{u}_i e_i^2\right]}{\sigma^2}Q^{-1}\right]\pi_\alpha, \\
\mathcal{B}_4 &= \frac{\alpha\left(\gamma-\alpha\right)}{\left(1-\alpha\right)^2}Q^{-1}\left[\left(I_p + \tilde{\Omega}Q^{-1}\right)\frac{E\left[\tilde{u}_i e_i^3\right]}{\sigma^2} - E\left[\tilde{u}_i\tilde{u}_i'Q^{-1}\tilde{u}_i e_i\right]\right. \\
&\qquad \left. - \frac{E\left[\tilde{u}_i\tilde{u}_i'\left(e_i^2-\sigma^2\right)\right]}{\sigma^2}Q^{-1}\Sigma\right].
\end{aligned}$$

The bias expression has a structure similar to that of the first order asymptotic variance. Analogously, the $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ components (like $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$) are positively related to α and vanish when $\alpha \to 0$. Analogously, these components are large and may overweigh the $\mathcal{B}_1$ component if the instruments are not strong so that Q is small.

Theorem (continued): Suppose assumptions 1–2 hold. The second order asymptotic bias of the Fuller estimator equals

$$\mathcal{B}^F = \mathcal{B} + \Delta\mathcal{B}_1^F,$$

where

$$\Delta\mathcal{B}_1^F = cQ^{-1}\Sigma.$$

Because $\mathcal{B}_1^F = -Q^{-1}\Sigma$, the leading bias component can be removed by setting $c = 1$, which is a usual recommendation. It was noticed by Morimune (1983) that the Fuller estimator with $c = 1$ does not fully remove the second order bias. In the case of one endogenous variable, the next component, $\mathcal{B}_2$, can be removed by further modifying the Fuller estimator (Morimune 1983), which does not though seem possible when $p > 1$ unless $\tilde{\Omega}$ is proportional to Q.

4 Special cases

Asymptotically balanced design of instruments means that the main diagonal of the projection matrix P is asymptotically homogeneous with values approaching α. See the theory and examples in Anatolyev and Yaskov (2017).

Corollary 1: Suppose assumptions 1–2 hold, and in addition $\lim P_{ii} = \alpha$ for all i. Then

$$\mathcal{B} = -\left[Q^{-1} + \frac{\alpha}{1-\alpha}Q^{-1}\tilde{\Omega}Q^{-1}\right]\Sigma.$$

Note that the matrix coefficients in square brackets are strictly positive definite.

In another special case *the structural error is mean, mean-square and mean-cube independent of the reduced form residual.* Then we have

Corollary 2: Suppose assumptions 1–2 hold, and in addition $E\left[e_i|\tilde{u}_i\right] = 0$, $E\left[e_i^2|\tilde{u}_i\right] = \sigma^2$ and $E\left[e_i^3|\tilde{u}_i\right] = v_3$. Then the bias components $\mathcal{B}_3$ and $\mathcal{B}_4$ become

$$\begin{aligned}\mathcal{B}_3 &= \frac{1}{1-\alpha}\frac{v_3}{\sigma^2}Q^{-1}\tilde{\Omega}Q^{-1}\pi_\alpha,\\ \mathcal{B}_4 &= 0.\end{aligned}$$

Finally, let *the structural and reduced form errors be jointly gaussian.* This leads to

Corollary 3: Suppose assumptions 1–2 hold, and in addition the vector $(e_i, u_i')'$ is mean zero normally distributed. Then

$$\mathcal{B}_3 = \mathcal{B}_4 = 0.$$

5 Closing remarks

The second order bias expressions can be used to analyze tendencies of estimators to be more or less biased in finite samples, as well as to construct analytical bias corrections. This is useful even though some moments may not exist, as is the case with LIML.[1] In this paper, we have derived second order biases of LIML and Fuller estimators in a homoskedastic environment. In heteroskedastic models, the LIML estimator ceases to be consistent. Hausman, Newey, Woutersen, Chao, and Swanson (2012) construct an alternative LIML-like estimator (as well as its Fuller modification) that uses jackknife ideas to reach consistency. Because of one-leave-outs, the asymptotic variance of the resulting estimators, being quite complicated in nature, largely contains only two terms, analogs of $\mathcal{V}_1$ and $\mathcal{V}_2$, even under error non-gaussianity. It is left for future research to investigate the second-order bias of these estimators and its structure.

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[1] The expectation of a higher order term in the expansion can be viewed as an approximate value of the bias in finite samples (e.g., Rothenberg 1984), similarly to how the asymptotic variance is viewed as an approximation to finite sample dispersion.

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A Online Appendix

Denote

$$\phi_1 = \sum_{i\neq j} P_{ij}^2, \quad \phi_2 = \sum_i (P_{ii}-\alpha)^2, \quad \varphi_1 = \sum_i \left(P_{ii}-\alpha\right) z_i.$$

Lemma. The following limits obtain: $\lim n^{-1}\phi_1 = \alpha\left(1-\gamma\right)$, $\lim n^{-1}\phi_2 = \alpha\left(\gamma-\alpha\right)$, and $\lim n^{-1}\Pi'\varphi = \pi_\alpha$.

Proof of Lemma. By properties of projection matrices, $\sum_j P_{ij}^2 = \sum_j P_{ij}P_{ji} = \left(P^2\right)_{ii} = P_{ii}$, hence $\sum_i\sum_j P_{ij}^2 = \sum_i P_{ii} = \mathrm{tr}(P) = \ell$. Then, first, $\phi_1 = \sum_i\sum_j P_{ij}^2 - \sum_i P_{ii}^2 = \ell - \sum_i P_{ii}^2$, while $\lim n^{-1}\ell = \alpha$ and $\lim n^{-1}\sum_i P_{ii}^2 = \alpha\gamma$ by assumptions 1 and 2, hence $\lim n^{-1}\phi_1 = \alpha\left(1-\gamma\right)$. Second, $\phi_2 = \sum_i P_{ii}^2 - 2\alpha\sum_i P_{ii} + \alpha^2 = \sum_i P_{ii}^2 - 2\alpha\ell$, and hence $\lim n^{-1}\phi_2 = \alpha\gamma - 2\alpha^2 + \alpha^2 = \alpha\left(\gamma-\alpha\right)$. Third, $\varphi = Z'D_P\iota_n - \alpha Z'\iota_n = Z'D_{P-\alpha I}\iota_n$, and hence $\lim n^{-1}\Pi'\varphi = \lim n^{-1}\Pi'Z'D_{P-\alpha I}\iota_n = \pi_\alpha$. $\square$

Proof of Theorem. Let us denote $\Gamma = \Sigma'/\sigma^2$ and $\tilde{U} = U - e\Gamma$. Note that $E\left[u_i\tilde{u}_i'\right] = \tilde{\Omega}$. The following correspondences in third and fourth moments will also be useful:

$$\begin{aligned}
E\left[u_iu_i'e_i\right] &= E\left[\tilde{u}_i\tilde{u}_i'e_i\right] + \Gamma'E\left[\tilde{u}_i'e_i^2\right] + E\left[\tilde{u}_ie_i^2\right]\Gamma + v_3\Gamma'\Gamma, \\
E\left[u_i\tilde{u}_i'e_i\right] &= E\left[\tilde{u}_i\tilde{u}_i'e_i\right] + \Gamma'E\left[\tilde{u}_i'e_i^2\right], \\
E\left[u_i'Q^{-1}\tilde{u}_ie_i\right] &= E\left[\tilde{u}_i'Q^{-1}\tilde{u}_ie_i\right] + \Gamma Q^{-1}E\left[\tilde{u}_ie_i^2\right], \\
E\left[u_iu_i'Q^{-1}\tilde{u}_ie_i\right] &= E\left[\tilde{u}_i\tilde{u}_i'Q^{-1}\tilde{u}_ie_i\right] + \Gamma'\Gamma Q^{-1}E\left[\tilde{u}_ie_i^3\right] + \Gamma'E\left[\tilde{u}_i'Q^{-1}\tilde{u}_ie_i^2\right] + E\left[\tilde{u}_i\tilde{u}_i'e_i^2\right]Q^{-1}\Gamma'.
\end{aligned}$$

In our derivations, we will use the elements of first order asymptotics such as $n^{-1}X'X = Q + \Omega + O_P\left(1/\sqrt{n}\right)$, $n^{-1}X'PX = Q + \alpha\Omega + O_P\left(1/\sqrt{n}\right)$, $n^{-1}X'e = \Sigma + O_P\left(1/\sqrt{n}\right)$, $n^{-1}X'Pe = \alpha\Sigma + O_P\left(1/\sqrt{n}\right)$, $n^{-1}e'\left(P-\alpha I\right)e = O_P\left(1/\sqrt{n}\right)$, etc. We will also extensively exploit that expectations of cross products of $\sum_{i\neq j}\eta_{ij}z_iu_j'$ and $\sum_{i\neq j}\eta_{ij}z_ie_j$ with $\sum_{m\neq k}\bar{\eta}_{mk}u_me_k$ are zero, and so are expectations of cross products of them and of $\sum_{i\neq j}\eta_{ij}u_iu_j'$ with $\sum_m\bar{\eta}_{mm}u_me_m$, where $\{\eta_{ij}\}_{i,j=1}^n$ and $\{\bar{\eta}_{ij}\}_{i,j=1}^n$ are arbitrary generic systems of constants. Unless otherwise indicated, I means I_n.

The LIML first order conditions are

$$\frac{1}{n}X'P\left(e-X\left(\hat{\theta}-\theta_0\right)\right) = \frac{n^{-1}\left(e-X\left(\hat{\theta}-\theta_0\right)\right)'P\left(e-X\left(\hat{\theta}-\theta_0\right)\right)}{n^{-1}\left(e-X\left(\hat{\theta}-\theta_0\right)\right)'\left(e-X\left(\hat{\theta}-\theta_0\right)\right)}\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right).$$

The ratio on the right side can be represented as

$$\alpha + \frac{n^{-1}e'\left(P-\alpha I\right)e - 2n^{-1}e'\left(P-\alpha I\right)X\left(\hat{\theta}-\theta_0\right) + \left(\hat{\theta}-\theta_0\right)'n^{-1}X'\left(P-\alpha I\right)X\left(\hat{\theta}-\theta_0\right)}{\sigma^2 + \left(n^{-1}e'e-\sigma^2\right) - 2n^{-1}e'X\left(\hat{\theta}-\theta_0\right) + \left(\hat{\theta}-\theta_0\right)'n^{-1}X'X\left(\hat{\theta}-\theta_0\right)}. \quad (3)$$

Hence, to first order,

$$\begin{aligned}
\frac{1}{n}X'P\left(e-X\left(\hat{\theta}-\theta_0\right)\right) &= \left(\alpha+\frac{1}{\sigma^2}\frac{e'\left(P-\alpha I\right)e}{n}\right)\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right) \\
&= \alpha\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right)+\frac{1}{\sigma^2}\frac{e'\left(P-\alpha I\right)e}{n}\frac{X'e}{n}+O_P\left(\frac{1}{n}\right) \\
&= \alpha\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right)+\frac{e'\left(P-\alpha I\right)e}{n}\Gamma'+O_P\left(\frac{1}{n}\right).
\end{aligned}$$

Thus, the first order expansion is

$$\begin{aligned}
\hat{\theta}-\theta_0 &= \left(\frac{X'\left(P-\alpha I\right)X}{n}\right)^{-1}\frac{\left(X-e\Gamma\right)'\left(P-\alpha I\right)e}{n}+O_P\left(\frac{1}{n}\right) \\
&= \left(1-\alpha\right)^{-1}Q^{-1}\frac{\left(X-e\Gamma\right)'\left(P-\alpha I\right)e}{n}+O_P\left(\frac{1}{n}\right). \qquad (4)
\end{aligned}$$

Next, expanding the ratio (3) further, we obtain on the right side of LIML first order conditions

$$\begin{aligned}
&\alpha\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right)+\frac{1}{\sigma^2}\left(1-\left(\frac{1}{\sigma^2}\frac{e'e}{n}-1\right)+\frac{2}{\sigma^2}\Sigma'\left(\hat{\theta}-\theta_0\right)+o_P\left(\frac{1}{\sqrt{n}}\right)\right) \\
&\times\left(\frac{e'\left(P-\alpha I\right)e}{n}-2\frac{e'\left(P-\alpha I\right)X}{n}\left(\hat{\theta}-\theta_0\right)+\left(\hat{\theta}-\theta_0\right)\left(1-\alpha\right)Q\left(\hat{\theta}-\theta_0\right)+o_P\left(\frac{1}{n}\right)\right) \\
&\times\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right)+o_P\left(\frac{1}{n}\right) \\
=\; &\alpha\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right)+\frac{1}{\sigma^2}\frac{e'\left(P-\alpha I\right)e}{n}\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta_0\right)\right) \\
&+\frac{1}{\sigma^2}\left(-\left(\frac{1}{\sigma^2}\frac{e'e}{n}-1\right)\frac{e'\left(P-\alpha I\right)e}{n}+\frac{2}{\sigma^2}\frac{e'\left(P-\alpha I\right)e}{n}\Sigma'\left(\hat{\theta}-\theta_0\right)\right)\Sigma \\
&+\frac{1}{\sigma^2}\left(-2\frac{e'\left(P-\alpha I\right)X}{n}\left(\hat{\theta}-\theta_0\right)+\left(1-\alpha\right)\left(\hat{\theta}-\theta_0\right)'Q\left(\hat{\theta}-\theta_0\right)\right)\Sigma+o_P\left(\frac{1}{n}\right).
\end{aligned}$$

This leads to the expansion to second order

$$\begin{aligned}
\left(1-\alpha\right)Q\left(\hat{\theta}-\theta_0\right) &= \frac{\left(X-e\Gamma\right)'\left(P-\alpha I\right)e}{n} \qquad (5) \\
&\quad-\left(\frac{X'\left(P-\alpha I\right)X}{n}-\left(1-\alpha\right)Q\right)\left(\hat{\theta}-\theta_0\right) \\
&\quad+\frac{1}{\sigma^2}\frac{e'\left(P-\alpha I\right)\Gamma'e}{n}\left(\frac{e'e}{n}-\sigma^2\right) \\
&\quad+2\Gamma'\frac{e'\left(P-\alpha I\right)\left(X-e\Gamma\right)}{n}\left(\hat{\theta}-\theta_0\right) \\
&\quad-\left(1-\alpha\right)\Gamma'\left(\hat{\theta}-\theta_0\right)'Q\left(\hat{\theta}-\theta_0\right) \\
&\quad-\frac{1}{\sigma^2}\frac{e'\left(P-\alpha I\right)e}{n}\left(\frac{X'e}{n}-\Sigma\right) \\
&\quad+\frac{1}{\sigma^2}\frac{e'\left(P-\alpha I\right)e}{n}\left(Q+\Omega\right)\left(\hat{\theta}-\theta_0\right)+o_P\left(\frac{1}{n}\right).
\end{aligned}$$

Because $2\Gamma'n^{-1}e'\left(P-\alpha I\right)\left(X-e\Gamma\right)-\left(1-\alpha\right)\Gamma'\left(\hat{\theta}-\theta_0\right)'Q=\Gamma'n^{-1}e'\left(P-\alpha I\right)\left(X-e\Gamma\right)+O_P\left(1/n\right)$, and because $\Gamma'\left(n^{-1}e'e-\sigma^2\right)-\left(n^{-1}X'e-\Sigma\right)=-n^{-1}\left(X-e\Gamma\right)'e$, and using the first order asymptotic expansion (4), we obtain

$$\begin{aligned}
(1-\alpha)\,Q\left(\hat{\theta}-\theta_0\right) &= \frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n} \\
&\quad -(1-\alpha)^{-1}\left(\frac{X'\,(P-\alpha I)\,X}{n}-(1-\alpha)\,Q\right)Q^{-1}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n} \\
&\quad +(1-\alpha)^{-1}\,\Gamma'\frac{e'\,(P-\alpha I)\,(X-e\Gamma)}{n}Q^{-1}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n} \\
&\quad -\frac{1}{\sigma^2}\frac{e'\,(P-\alpha I)\,e}{n}\frac{(X-e\Gamma)'\,e}{n} \\
&\quad +\frac{1}{\sigma^2}\,(1-\alpha)^{-1}\left(I+\Omega Q^{-1}\right)\frac{e'\,(P-\alpha I)\,e}{n}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n} \\
&\quad +o_P\left(\frac{1}{n}\right).
\end{aligned}$$

Hence, apart from the prefactor $(1-\alpha)\,Q$, there are four terms whose limits (after multiplication by n) belong to the second order bias of LIML:

$$\begin{aligned}
b_1 &= (1-\alpha)^{-1}\,\Gamma' E\left[\frac{e'\,(P-\alpha I)\,(X-e\Gamma)}{n}Q^{-1}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n}\right], \\
b_2 &= -(1-\alpha)^{-1}\,E\left[\frac{X'\,(P-\alpha I)\,X}{n}Q^{-1}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n}\right], \\
b_3 &= -\frac{1}{\sigma^2}E\left[\frac{e'\,(P-\alpha I)\,e}{n}\frac{(X-e\Gamma)'\,e}{n}\right], \\
b_4 &= \frac{1}{\sigma^2}\,(1-\alpha)^{-1}\left(I_p+\Omega Q^{-1}\right)E\left[\frac{e'\,(P-\alpha I)\,e}{n}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n}\right].
\end{aligned}$$

Consider each term in turn. In the first term b_1,

$$\begin{aligned}
& E\left[\frac{e'\,(P-\alpha I)\,(X-e\Gamma)}{n}Q^{-1}\frac{(X-e\Gamma)'\,(P-\alpha I)\,e}{n}\right] \\
={}& E\left[\left((1-\alpha)\,\frac{\Pi'Z'e}{n}+\frac{\tilde{U}'\,(P-\alpha I)\,e}{n}\right)'Q^{-1}\left((1-\alpha)\,\frac{\Pi'Z'e}{n}+\frac{\tilde{U}'\,(P-\alpha I)\,e}{n}\right)\right] \\
={}& (1-\alpha)^2\,E\left[\frac{e'Z\Pi}{n}Q^{-1}\frac{\Pi'Z'e}{n}\right]+E\left[\left(\frac{\tilde{U}'\,(P-\alpha I)\,e}{n}\right)'Q^{-1}\frac{\tilde{U}'\,(P-\alpha I)\,e}{n}\right] \\
& +2\,(1-\alpha)\,E\left[\left(\frac{\Pi'Z'e}{n}\right)'Q^{-1}\frac{\tilde{U}'\,(P-\alpha I)\,e}{n}\right]
\end{aligned}$$

For the first part,

$$n^2 E\left[\frac{e'Z\Pi}{n}Q^{-1}\frac{\Pi'Z'e}{n}\right]=\sigma^2 E\left[\mathrm{tr}(Q^{-1}\Pi'Z'Z\Pi)\right]=n\sigma^2 p+o\left(n\right).$$

For the second part,

$$
\begin{aligned}
& n^2 E\left[\left(\frac{\tilde{U}'(P-\alpha I)e}{n}\right)' Q^{-1}\frac{\tilde{U}'(P-\alpha I)e}{n}\right] \\
= \; & E\left[\left(\sum_{i\neq j} P_{ij}\tilde{u}_i e_j + \sum_i (P_{ii}-\alpha)\tilde{u}_i e_i\right)' Q^{-1}\left(\sum_{m\neq k} P_{mk}\tilde{u}_m e_k + \sum_m (P_{mm}-\alpha)\tilde{u}_m e_m\right)\right] \\
= \; & \sum_{i\neq j} P_{ij}^2 E\left[\tilde{u}_i' e_j Q^{-1}\tilde{u}_i e_j\right] + \sum_{i\neq j} P_{ij}^2 E\left[\tilde{u}_i' e_j Q^{-1}\tilde{u}_j e_i\right] \\
& + \sum_{i\neq m} (P_{mm}-\alpha)(P_{ii}-\alpha) E\left[\tilde{u}_i' e_i Q^{-1}\tilde{u}_m e_m\right] + \sum_i (P_{ii}-\alpha)^2 E\left[\tilde{u}_i' e_i Q^{-1}\tilde{u}_i e_i\right] \\
= \; & \phi_1\sigma^2 \mathrm{tr}\left(\tilde{\Omega}Q^{-1}\right) + \phi_2 \mathrm{tr}\left(E\left[\tilde{u}_i\tilde{u}_i' e_i^2\right]Q^{-1}\right).
\end{aligned}
$$

For the third part,

$$
\begin{aligned}
& n^2 E\left[\left(\frac{\Pi' Z' e}{n}\right)' Q^{-1}\frac{\tilde{U}'(P-\alpha I)e}{n}\right] \\
= \; & E\left[\sum_i z_i' e_i \Pi Q^{-1}\left(\sum_{m\neq k} P_{mk}\tilde{u}_m e_k + \sum_m (P_{mm}-\alpha)\tilde{u}_m e_m\right)\right] \\
= \; & \sum_m (P_{mm}-\alpha) z_m' \Pi Q^{-1} E\left[\tilde{u}_m e_m^2\right] \\
= \; & \varphi' \Pi Q^{-1} E\left[\tilde{u}_m e_m^2\right].
\end{aligned}
$$

The three parts together imply

$$
\begin{aligned}
& n^2 E\left[\frac{e'(P-\alpha I)(X-e\Gamma)}{n} Q^{-1}\frac{(X-e\Gamma)'(P-\alpha I)e}{n}\right] \\
= \; & (1-\alpha)^2 n\sigma^2 p + \phi_1\sigma^2 \mathrm{tr}\left(\tilde{\Omega}Q^{-1}\right) + \phi_2 \mathrm{tr}\left(E\left[\tilde{u}_i\tilde{u}_i' e_i^2\right]Q^{-1}\right) + 2(1-\alpha)\varphi'\Pi Q^{-1}E\left[\tilde{u}_m e_m^2\right] \\
& + o(n),
\end{aligned}
$$

and hence, taking limits and using Lemma A2,

$$
\begin{aligned}
\lim n b_1 \; = \; & (1-\alpha)\Sigma p + 2\sigma^{-2}\Sigma E\left[\tilde{u}_m' e_m^2\right] Q^{-1}\pi_\alpha + (1-\alpha)^{-1}\alpha(1-\gamma)\,\mathrm{tr}\left(\tilde{\Omega}Q^{-1}\right)\Sigma \\
& + (1-\alpha)^{-1}\alpha(\gamma-\alpha)\sigma^{-2}\mathrm{tr}\left(E\left[\tilde{u}_i\tilde{u}_i' e_i^2\right]Q^{-1}\right)\Sigma.
\end{aligned}
$$

In the second term b_2,

$$
\begin{aligned}
& E\left[\frac{X'\left(P-\alpha I\right)X}{n}Q^{-1}\frac{\left(X-e\Gamma\right)'\left(P-\alpha I\right)e}{n}\right] \\
= \; & E\left[\left(\frac{U'\left(P-\alpha I\right)U}{n}+\left(1-\alpha\right)\left(\frac{\Pi'Z'U}{n}+\frac{U'Z\Pi}{n}\right)\right)Q^{-1}\left(\left(1-\alpha\right)\frac{\Pi'Z'e}{n}+\frac{\tilde{U}'\left(P-\alpha I\right)e}{n}\right)\right] \\
= \; & \left(1-\alpha\right)E\left[\frac{U'\left(P-\alpha I\right)U}{n}Q^{-1}\frac{\Pi'Z'e}{n}\right]+E\left[\frac{U'\left(P-\alpha I\right)U}{n}Q^{-1}\frac{\tilde{U}'\left(P-\alpha I\right)e}{n}\right] \\
& +\left(1-\alpha\right)^2E\left[\left(\frac{\Pi'Z'U}{n}+\frac{U'Z\Pi}{n}\right)Q^{-1}\frac{\Pi'Z'e}{n}\right] \\
& +\left(1-\alpha\right)E\left[\left(\frac{\Pi'Z'U}{n}+\frac{U'Z\Pi}{n}\right)Q^{-1}\frac{\tilde{U}'\left(P-\alpha I\right)e}{n}\right]
\end{aligned}
$$

For the first part,

$$
\begin{aligned}
n^2E\left[\frac{U'\left(P-\alpha I\right)U}{n}Q^{-1}\frac{\Pi'Z'e}{n}\right] &= E\left[\left(\sum_{i\neq j}P_{ij}u_iu_j'+\sum_i\left(P_{ii}-\alpha\right)u_iu_i'\right)Q^{-1}\Pi'\sum_m z_me_m\right] \\
&= \sum_i\left(P_{ii}-\alpha\right)E\left[u_iu_i'e_i\right]Q^{-1}\Pi'z_i \\
&= \left(E\left[\tilde{u}_i\tilde{u}_i'e_i\right]+\Gamma'E\left[\tilde{u}_i'e_i^2\right]+E\left[\tilde{u}_ie_i^2\right]\Gamma+v_3\Gamma'\Gamma\right)Q^{-1}\Pi'\varphi.
\end{aligned}
$$

For the second part,

$$
\begin{aligned}
& n^2E\left[\frac{U'\left(P-\alpha I\right)U}{n}Q^{-1}\frac{\tilde{U}'\left(P-\alpha I\right)e}{n}\right] \\
= \; & E\left[\left(\sum_{i\neq j}P_{ij}u_iu_j'+\sum_i\left(P_{ii}-\alpha\right)u_iu_i'\right)Q^{-1}\left(\sum_{m\neq k}P_{mk}\tilde{u}_me_k+\sum_m\left(P_{mm}-\alpha\right)\tilde{u}_me_m\right)\right] \\
= \; & \sum_{i\neq j}P_{ij}^2E\left[u_iu_j'Q^{-1}\tilde{u}_ie_j\right]+\sum_{i\neq j}P_{ij}^2E\left[u_iu_j'Q^{-1}\tilde{u}_je_i\right] \\
& +\sum_i\sum_{m\neq k}P_{mk}\left(P_{ii}-\alpha\right)E\left[u_iu_i'Q^{-1}\tilde{u}_me_k\right]+\sum_{i\neq j}\sum_m\left(P_{mm}-\alpha\right)P_{ij}E\left[u_iu_j'Q^{-1}\tilde{u}_me_m\right] \\
& +\sum_{i\neq m}\left(P_{mm}-\alpha\right)\left(P_{ii}-\alpha\right)E\left[u_iu_i'Q^{-1}\tilde{u}_me_m\right]+\sum_i\left(P_{ii}-\alpha\right)^2E\left[u_iu_i'Q^{-1}\tilde{u}_ie_i\right] \\
= \; & \phi_1\tilde{\Omega}Q^{-1}\Sigma+\phi_1\mathrm{tr}\left(\tilde{\Omega}Q^{-1}\right)\Sigma \\
& +\phi_2\left(E\left[\tilde{u}_i\tilde{u}_i'Q^{-1}\tilde{u}_ie_i\right]+\Gamma'\Gamma Q^{-1}E\left[\tilde{u}_ie_i^3\right]+\Gamma'E\left[\tilde{u}_i'Q^{-1}\tilde{u}_ie_i^2\right]+E\left[\tilde{u}_i\tilde{u}_i'e_i^2\right]Q^{-1}\Gamma'\right).
\end{aligned}
$$

For the third part,

$$
\begin{aligned}
n^2E\left[\left(\frac{\Pi'Z'U}{n}+\frac{U'Z\Pi}{n}\right)Q^{-1}\frac{\Pi'Z'e}{n}\right] &= \Pi'\sum_i z_iE\left[u_i'e_i\right]Q^{-1}\Pi'z_i+\sum_i E\left[u_ie_i\right]z_i'\Pi Q^{-1}\Pi'z_i \\
&= \Pi'Z'Z\Pi Q^{-1}\Sigma+\Sigma\mathrm{tr}\left(\Pi'Z'Z\Pi Q^{-1}\right).
\end{aligned}
$$

For the fourth part,

$$
\begin{aligned}
& n^2 E\left[\left(\frac{\Pi' Z' U}{n}+\frac{U' Z \Pi}{n}\right) Q^{-1} \frac{\tilde{U}'(P-\alpha I) e}{n}\right] \\
= \;& E\left[\left(\Pi' \sum_i z_i u_i' + \sum_i u_i z_i' \Pi\right) Q^{-1}\left(\sum_{m \neq k} P_{mk} \tilde{u}_m e_k + \sum_m (P_{mm}-\alpha) \tilde{u}_m e_m\right)\right] \\
= \;& \Pi' \varphi\left(E\left[\tilde{u}_i' Q^{-1} \tilde{u}_i e_i\right] + \Gamma Q^{-1} E\left[\tilde{u}_i e_i^2\right]\right) + \left(E\left[\tilde{u}_i \tilde{u}_i' e_i\right] + \Gamma' E\left[\tilde{u}_i' e_i^2\right]\right) Q^{-1} \Pi' \varphi .
\end{aligned}
$$

The four parts together imply

$$
\begin{aligned}
& n^2 E\left[\frac{X'(P-\alpha I) X}{n} Q^{-1} \frac{(X-e\Gamma)'(P-\alpha I) e}{n}\right] \\
= \;& (1-\alpha)\left(E\left[\tilde{u}_i \tilde{u}_i' e_i\right] + \Gamma' E\left[\tilde{u}_i' e_i^2\right] + E\left[\tilde{u}_i e_i^2\right] \Gamma + v_3 \Gamma' \Gamma\right) Q^{-1} \Pi' \varphi \\
& + \phi_1 \tilde{\Omega} Q^{-1} \Sigma + \phi_1 \operatorname{tr}\left(\tilde{\Omega} Q^{-1}\right) \Sigma \\
& + \phi_2\left(E\left[\tilde{u}_i \tilde{u}_i' Q^{-1} \tilde{u}_i e_i\right] + \Gamma' \Gamma Q^{-1} E\left[\tilde{u}_i e_i^3\right] + \Gamma' E\left[\tilde{u}_i' Q^{-1} \tilde{u}_i e_i^2\right] + E\left[\tilde{u}_i \tilde{u}_i' e_i^2\right] Q^{-1} \Gamma'\right) \\
& + (1-\alpha)^2\left(\Pi' Z' Z \Pi Q^{-1} \Sigma + \Sigma \operatorname{tr}\left(\Pi' Z' Z \Pi Q^{-1}\right)\right) \\
& + (1-\alpha)\left(\Pi' \varphi\left(E\left[\tilde{u}_i' Q^{-1} \tilde{u}_i e_i\right] + \Gamma Q^{-1} E\left[\tilde{u}_i e_i^2\right]\right) + \left(E\left[\tilde{u}_i \tilde{u}_i' e_i\right] + \Gamma' E\left[\tilde{u}_i' e_i^2\right]\right) Q^{-1} \Pi' \varphi\right) \\
& + o(n) .
\end{aligned}
$$

and hence, taking limits and using Lemma A2,

$$
\begin{aligned}
\lim n b_2 = \;& -(1-\alpha)(1+p) \Sigma - (1-\alpha)^{-1} \alpha (1-\gamma)\left(\tilde{\Omega} Q^{-1} + \operatorname{tr}\left(\tilde{\Omega} Q^{-1}\right)\right) \Sigma \\
& - \left(2E\left[\tilde{u}_i \tilde{u}_i' e_i\right] + 2\Gamma' E\left[\tilde{u}_i' e_i^2\right] + E\left[\tilde{u}_i e_i^2\right] \Gamma + v_3 \Gamma' \Gamma\right) Q^{-1} \pi_\alpha \\
& - \left(E\left[\tilde{u}_i' Q^{-1} \tilde{u}_i e_i\right] + \Gamma Q^{-1} E\left[\tilde{u}_i e_i^2\right]\right) \pi_\alpha - (1-\alpha)^{-1} \alpha (\gamma-\alpha) E\left[\tilde{u}_i \tilde{u}_i' Q^{-1} \tilde{u}_i e_i\right] \\
& - (1-\alpha)^{-1} \alpha (\gamma-\alpha)\left(\Gamma' \Gamma Q^{-1} E\left[\tilde{u}_i e_i^3\right] + \Gamma' E\left[\tilde{u}_i' Q^{-1} \tilde{u}_i e_i^2\right] + E\left[\tilde{u}_i \tilde{u}_i' e_i^2\right] Q^{-1} \Gamma'\right) .
\end{aligned}
$$

In the third term b_3,

$$
\begin{aligned}
& n^2 E\left[\frac{e'(P-\alpha I) e}{n} \frac{(X-e\Gamma)' e}{n}\right] \\
= \;& E\left[\left(\sum_{i \neq j} P_{ij} e_i e_j + \sum_i (P_{ii}-\alpha) e_i^2\right)\left(\Pi' \sum_m z_m e_m + \sum_m \tilde{u}_m e_m\right)\right] \\
= \;& \sum_{i \neq m} (P_{ii}-\alpha) \sigma^2 E\left[\Pi' z_m e_m + \tilde{u}_m e_m\right] + \sum_i (P_{ii}-\alpha) E\left[e_i^2\left(\Pi' z_i e_i + \tilde{u}_i e_i\right)\right] \\
= \;& \Pi' \varphi v_3 + o(1),
\end{aligned}
$$

also using assumption 1. Hence, taking limits and using Lemma A3,

$$
\lim n b_3 = -\sigma^{-2} v_3 \pi_\alpha .
$$

In the fourth term b_4, similarly,

$$\begin{aligned}
& n^2 E\left[\frac{e'(P-\alpha I)e}{n}\left((1-\alpha)\frac{\Pi' Z' e}{n}+\frac{\tilde{U}'(P-\alpha I)e}{n}\right)\right] \\
= \ & (1-\alpha)n^2 E\left[\frac{e'(P-\alpha I)e}{n}\frac{\Pi' Z' e}{n}\right]+n^2 E\left[\frac{e'(P-\alpha I)e}{n}\frac{\tilde{U}'(P-\alpha I)e}{n}\right] \\
= \ & (1-\alpha)\Pi'\varphi v_3+\phi_2 E\left[\tilde{u}_i e_i^3\right].
\end{aligned}$$

hence, taking limits and using Lemma A3,

$$\lim nb_4=\sigma^{-2}v_3\left(I_p+\Omega Q^{-1}\right)\pi_\alpha+\sigma^{-2}(1-\alpha)^{-1}\alpha(\gamma-\alpha)\left(I_p+\Omega Q^{-1}\right)E\left[\tilde{u}_i e_i^3\right].$$

Gathering the pieces and adding them together, we obtain the expression as in the Theorem.

Now we proceed to the Fuller estimator. Note that

$$\mathring{\alpha}=\tilde{\alpha}-(1-\alpha)\frac{c}{n}+o\left(\frac{1}{n}\right),$$

also using assumption 1. Therefore, the Fuller estimator's first order conditions are, to second order,

$$\begin{aligned}
\frac{1}{n}X'P\left(e-X\left(\hat{\theta}-\theta\right)\right) &= \left(\tilde{\alpha}-(1-\alpha)\frac{c}{n}+o\left(\frac{1}{n}\right)\right)\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta\right)\right) \\
&= \tilde{\alpha}\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta\right)\right)-(1-\alpha)\frac{c}{n}\cdot\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta\right)\right) \\
&\quad +o_P\left(\frac{1}{n}\right).
\end{aligned}$$

The extra term adds to the right hand side of the expansion (5) the term

$$(1-\alpha)\frac{c}{n}\cdot\frac{1}{n}X'\left(e-X\left(\hat{\theta}-\theta\right)\right)=(1-\alpha)\frac{c}{n}\Sigma+o_P\left(\frac{1}{n}\right),$$

which contributes to the bias an additional component equal to $cQ^{-1}\Sigma$. □

Proof of Corollary 1: In the special case of asymptotically balanced design of P, we have $\gamma=\alpha$, $\pi_\alpha=0$, andt he bias components $\mathcal{B}_3$ and $\mathcal{B}_4$ vanish. The rest is straightforward. □

Proof of Corollary 2: Error moment independence implies $E\left[\tilde{u}_i e_i\right]=0$, $E\left[\tilde{u}_i\tilde{u}_i' e_i\right]=0$, $E\left[\tilde{u}_i e_i^2\right]=0$, $E\left[\tilde{u}_i e_i^3\right]=0$, $E\left[\tilde{u}_i\tilde{u}_i' Q^{-1}\tilde{u}_i e_i\right]=0$ and $E\left[\tilde{u}_i\tilde{u}_i' Q^{-1}\left(e_i^2-\sigma^2\right)\right]=0$. Also,

$$\begin{aligned}
E\left[u_i u_i' e_i\right] &= \frac{v_3}{\sigma^4}\Sigma\Sigma', \\
E\left[u_i u_i' e_i^2\right] &= \sigma^2\tilde{\Omega}+\frac{v_4}{\sigma^4}\Sigma\Sigma', \\
E\left[u_i u_i' Q^{-1}u_i e_i\right] &= \operatorname{tr}\left(\tilde{\Omega}Q^{-1}\right)\Sigma+2\tilde{\Omega}Q^{-1}\Sigma+\frac{v_4}{\sigma^6}\left(\Sigma' Q^{-1}\Sigma\right)\Sigma.
\end{aligned}$$

The rest is straightforward. □

Proof of Corollary 3: Error normality implies $v_3=0$, $v_4=3\sigma^4$. The rest is straightforward. □