Method-of-moments estimation and choice of instruments

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Technical appendix

1 Basic expectations

If

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix} \right),$$

then

$$\begin{aligned} E\left[\exp\left(u_1\right)\right] &= \exp\left(\mu_1 + .5\omega_{11}\right) \\ E\left[\exp\left(u_1\right)u_2\right] &= \left(\mu_2 + \omega_{12}\right)\exp\left(\mu_1 + .5\omega_{11}\right) \\ E\left[\exp\left(u_1\right)u_2^2\right] &= \left(\left(\mu_2 + \omega_{12}\right)^2 + \omega_{22}\right)\exp\left(\mu_1 + .5\omega_{11}\right) \end{aligned}$$

In particular, if $u_i = \gamma_i U$, $i = 1, 2$, and $U \sim N\left(E_U, V_U\right)$, then the above formulae may be applied with $\mu_i = \gamma_i E_U$, $\omega_{ii} = \gamma_i V_U \gamma_i'$, $i = 1, 2$, and $\omega_{12} = \gamma_1 V_U \gamma_2'$.

2 Model details

Recall that $x_{1,t}$ is the one period rate of return, $x_{2,t}$ is one period consumption growth, and the $(q+1)$-period problem is rewritten as

$$E\left[\beta^{q+1} x_{1,t+1} x_{1,t+2} \cdots x_{1,t+q+1} x_{2,t+1}^{\alpha} x_{2,t+2}^{\alpha} \cdots x_{2,t+q+1}^{\alpha} - 1 | I_t\right] = 0.$$

The one-period model is

$$E\left[\beta x_{1,t+1} x_{2,t+1}^{\alpha} - 1 | I_t\right] = 0,$$

with the conditional moment function

$$\mu_{t+1} = \beta x_{1,t+1} x_{2,t+1}^{\alpha} - 1.$$

The unconditional moment function is

$$m_t = z_t\left(\beta x_{1,t+1} x_{2,t+1}^{\alpha} - 1\right),$$

with the first derivative

$$m_{\theta t} = z_t x_{1,t+1} x_{2,t+1}^{\alpha}\left(1, \beta \log(x_{2,t+1})\right).$$

The two-period model is

$$E\left[\beta^2 x_{1,t+1} x_{1,t+2} x_{2,t+1}^{\alpha} x_{2,t+2}^{\alpha} - 1 | I_t\right] = 0,$$

with the conditional moment function

$$\mu_{t+2} = \beta^2 x_{1,t+1} x_{1,t+2} x_{2,t+1}^{\alpha} x_{2,t+2}^{\alpha} - 1.$$

The unconditional moment function is

$$m_t = z_t \left(\beta^2 x_{1,t+1} x_{1,t+2} x_{2,t+1}^{\alpha} x_{2,t+2}^{\alpha} - 1\right),$$

with the first derivative

$$m_{\theta t} = \beta z_t x_{1,t+1} x_{1,t+2} x_{2,t+1}^{\alpha} x_{2,t+2}^{\alpha} \left(2, \beta \log(x_{2,t+1} x_{2,t+2})\right).$$

The vector of instruments is

$$z_t = \left(x_{1,t}, x_{1,t-1}, \cdots, x_{1,t-nl_1+1}, x_{2,t}, x_{2,t-1}, \cdots\ x_{2,t-nl_2+1}, 1\right)'.$$

The vector $x_t \equiv (x_{1,t}, x_{2,t})'$ is lognormally distributed, and the law of motion for $X_t \equiv (\log x_{1,t}, \log x_{2,t})'$ is stationary VAR(1) with normal innovations $U_t \equiv (u_{1,t}, u_{2,t})'$:

$$X_t = \lambda + \Phi X_{t-1} + U_t, \quad U_t \sim IID\ N\left(0, V_U\right),$$

where

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \quad V_U = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Denote

$$\begin{aligned} E_X &\equiv E\left[X_t\right] = \left(I_2 - \Phi\right)^{-1} \lambda, \\ V_X &\equiv V\left[X_t\right] = V_U + \Phi V_U \Phi' + \Phi^2 V_U \Phi'^2 + \cdots. \end{aligned}$$

Estimation of the law of motion was performed subject to constraints imposed by the conditional moment restriction. The constraints are (see Hansen and Singleton, 1982, formulas (4.5))

$$\begin{aligned} 0 &= (1, \alpha)\, \Phi \\ 0 &= \log \beta + (1, \alpha)\, \lambda + \frac{(1, \alpha)\, V_U\, (1, \alpha)'}{2} \end{aligned}$$

Essentially, only one constraint on the parameters of the law of motion is imposed; the other two restrictions determine the deep parameters β and α. Calibration using the data from the Hansen–Heaton–Ogaki (1994) GMM package yields

$$\begin{aligned} \lambda_1 &= 0.01571,\ \lambda_2 = 0.003291, \\ \phi_{11} &= 0.04636,\ \phi_{12} = 0.01435,\ \phi_{21} = 0.3935,\ \phi_{22} = 0.1218, \\ \sigma_1^2 &= 0.006349,\ \sigma_2^2 = 3.221 \times 10^{-5},\ \sigma_{12} = 0.0001086, \\ \beta &= 0.9817,\ \alpha = -0.1178. \end{aligned}$$

For the one-period model, the DGP is such that the moment function is conditionally homoskedastic with the variance

$$\sigma_m^2 = \exp\left((1, \alpha)\, V_U\, (1, \alpha)'\right) - 1.$$

In the calibrated model $\sigma_m^2 = 0.006344$. For the two-period model, the DGP is such that the moment function is conditionally homoskedastic and homoautocorrelated, with the variance and covariance, respectively,

$$\begin{aligned} \sigma_m^2 &= \exp\left(2\, (1, \alpha)\, V_U\, (1, \alpha)'\right) - 1, \\ \gamma_m &= \exp\left((1, \alpha)\, V_U\, (1, \alpha)'\right) - 1. \end{aligned}$$

This implies the autocorrelation coefficient and implied MA(1) coefficient, respectively,

$$\begin{aligned} \rho_m &= \left(\exp\left((1, \alpha)\, V_U\, (1, \alpha)'\right) + 1\right)^{-1}, \\ \varrho_m &= \frac{1}{2}\left(\rho_m^{-1} - \sqrt{\rho_m^{-2} - 4}\right). \end{aligned}$$

In the calibrated model $\sigma_m^2 = 0.012728$, $\rho_m = 0.4984$, and $\varrho_m = 0.9235$. As one can see, the serial correlation is pretty severe. It is worth noting that the conditional moment being time invariant is a result of the lognormal specification. It is very fortunate as it simplifies further computations and makes possible analytical derivation of optimal instruments and efficiency bounds (see Anatolyev, 2003).

3 Derivative expectations

Denote

$$V_\Sigma(k) \equiv V_U + \Phi V_U \Phi' + \ldots + \Phi^{k-1} V_U \Phi^{k-1\prime} + \Phi^k V_U \Phi^{k\prime}.$$

In all subsequent derivations, we make use of the fact that $(1,\alpha)\Phi = 0$ and $\log\beta + (1,\alpha) E_X + .5(1,\alpha) V_U (1,\alpha)' = 0$. Let $\rho \geq 0$ be an integer, $\pi = 0,1$, $\kappa = 1,2$, and $i_1, i_2 = 1,2$; $j_1 = 0,1,\cdots,nl_{i_1}-1$; $j_2 = 0,1,\cdots,nl_{i_2}-1$. Let "$\circ$" designate index when its value does not matter. It is easily seen that for $j_1 \geq j_2$

$$S^\rho \equiv E\left[x_{1,t+1}^\rho x_{2,t+1}^{\rho\alpha}\right] = \exp(\rho(1,\alpha)E_X) \times E\left[\exp(\rho(1,\alpha)(X_t - E_X))\right]$$

$$\begin{aligned} T_{i_1,j_1}^\rho &\equiv E\left[x_{i_1,t-j_1} x_{1,t+1}^\rho x_{2,t+1}^{\rho\alpha}\right] \\ &= \exp\left((e_{i_1} + \rho(1,\alpha))E_X + .5\rho^2(1,\alpha)V_U(1,\alpha)'\right) \times E\left[\exp(e_{i_1}(X_t - E_X))\right] \end{aligned}$$

$$\begin{aligned} P_{i_1,j_1,i_2,j_2}^\rho &\equiv E\left[x_{i_1,t-j_1} x_{i_2,t-j_2} x_{1,t+1}^\rho x_{2,t+1}^{\rho\alpha}\right] \\ &= \exp\left((e_{i_1} + e_{i_2} + \rho(1,\alpha))E_X + .5\rho^2(1,\alpha)V_U(1,\alpha)' + .5 e_{i_2} V_\Sigma(j_1 - j_2 - 1) e_{i_2}'\right) \\ &\quad \times E\left[\exp\left(\left(e_{i_1} + e_{i_2}\Phi^{j_1 - j_2}\right)(X_t - E_X)\right)\right] \end{aligned}$$

On the basis of "Basic expectations" we compute the following moments:

$$\begin{aligned} A_{i_1,j_1}^{\pi\rho}(\kappa) &\equiv E\left[x_{i_1,t-j_1}^\pi x_{1,t+1}^\rho x_{1,t+2}^\rho x_{2,t+1}^{\rho\alpha} x_{2,t+2}^{\rho\alpha} \log^\kappa(x_{2,t+1}x_{2,t+2})\right], \\ W_{i_1,j_1}^{\pi\rho}(\kappa) &\equiv E\left[x_{i_1,t-j_1}^\pi x_{1,t+1}^\rho x_{2,t+1}^{\rho\alpha} \log^\kappa(x_{2,t+1})\right]. \end{aligned}$$

Denote

$$\begin{aligned} c_1(j) &\equiv e_2 V_U (1,\alpha)' + e_2\left(I_2 - \Phi^{j+1}\right)E_X, \\ c_2(j) &\equiv e_2(2I_2 + \Phi)V_U(1,\alpha)' + e_2\left(2I_2 - (I_2 + \Phi)\Phi^{j+1}\right)E_X. \end{aligned}$$

Then

$$\begin{aligned} \beta^2 A_{i_1,j_1}^{11}(1) &= c_2(j_1) E\left[\exp(e_{i_1}X_t)\right] + E\left[\exp(e_{i_1}X_t)(v_e(j_1+1)X_t)\right] \\ \beta^2 A_{i_1,j_1}^{11}(2) &= \left(c_2(j_1)^2 + e_2\left(V_U + (I_2+\Phi)V_\Sigma(j_1)(I_2+\Phi')\right)e_2'\right) E\left[\exp(e_{i_1}X_t)\right] \\ &\quad + 2c_2(j_1) E\left[\exp(e_{i_1}X_t)(v_e(j_1+1)X_t)\right] + E\left[\exp(e_{i_1}X_t)(v_e(j_1+1)X_t)^2\right] \end{aligned}$$

$$\begin{aligned} \beta A_{\circ,\circ}^{01}(1) &= c_1(0) E\left[\exp((1,\alpha)X_t)\right] + E\left[\exp((1,\alpha)X_t)(e_2(I_2+\Phi)X_t)\right] \\ \beta A_{\circ,\circ}^{01}(2) &= \left(c_1(0)^2 + e_2 V_U e_2'\right) E\left[\exp((1,\alpha)X_t)\right] \\ &\quad + 2c_1(0) E\left[\exp((1,\alpha)X_t)(v_e(0)X_t)\right] + E\left[\exp((1,\alpha)X_t)(v_e(0)X_t)^2\right] \end{aligned}$$

$$\begin{aligned} \beta W_{i_1,j_1}^{11}(1) &= c_1(j_1) E\left[\exp(e_{i_1}X_t)\right] + E\left[\exp(e_{i_1}X_t)\left(e_2\Phi^{j_1+1}X_t\right)\right] \\ \beta W_{i_1,j_1}^{11}(2) &= \left(c_1(j_1)^2 + e_2 V_\Sigma(j_1) e_2'\right) E\left[\exp(e_{i_1}X_t)\right] \\ &\quad + 2c_1(j_1) E\left[\exp(e_{i_1}X_t)\left(e_2\Phi^{j_1+1}X_t\right)\right] + E\left[\exp(e_{i_1}X_t)\left(e_2\Phi^{j_1+1}X_t\right)^2\right] \\ W_{\circ,\circ}^{0\rho}(\kappa) &= E\left[\exp(\rho(1,\alpha)X_t)(e_2X_t)^\kappa\right] \end{aligned}$$

where $v_e(j) \equiv e_2(I_2+\Phi)\Phi^j$.

We also need

$$\Gamma_0 = E\left[z_t z_t'\right] = \begin{bmatrix} \mathrm{P}_{0,1,1} & \mathrm{P}_{0,1,2} & \mathrm{T}_{0,1} \\ \mathrm{P}_{0,2,1} & \mathrm{P}_{0,2,2} & \mathrm{T}_{0,2} \\ \mathrm{T}_{0,1}' & \mathrm{T}_{0,2}' & 1 \end{bmatrix}, \quad \Gamma_1 = E\left[z_t z_{t-1}'\right] = \begin{bmatrix} \mathrm{P}_{1,1,1} & \mathrm{P}_{1,1,2} & \mathrm{T}_{0,1} \\ \mathrm{P}_{1,2,1} & \mathrm{P}_{1,2,2} & \mathrm{T}_{0,2} \\ \mathrm{T}_{1,1}' & \mathrm{T}_{1,2}' & 1 \end{bmatrix}$$

where

$$\begin{aligned}\mathrm{P}_{0,i_1,i_2} &= \left\| P^0_{i_1,j_1,i_2,j_2} \right\|_{j_1=0,\cdots,nl_{i_1}-1,\ j_2=0,\cdots,nl_{i_2}-1} \qquad \mathrm{T}_{0,i_1} = \left\| T^0_{i_1,j_1} \right\|_{j_1=0,\cdots,nl_{i_1}-1} \\ \mathrm{P}_{1,i_1,i_2} &= \left\| P^0_{i_1,j_1,i_2,j_2} \right\|_{j_1=0,\cdots,nl_{i_1}-1,\ j_2=1,\cdots,nl_{i_2}} \qquad \mathrm{T}_{1,i_1} = \left\| T^0_{i_1,j_1} \right\|_{j_1=1,\cdots,nl_{i_1}}\end{aligned}$$

4 Derivation of optimal instrument

Because of conditional homoskedasticity and homoautocorrelatedness, it is possible to derive the optimal instrument, i.e. the one that allows attaining the GMM/EL efficiency bound, the greatest lower bound for the asymptotic variance of GMM/EL estimators (Hansen, 1985).

For the one-period problem, because of conditional homoskedasticity, the optimal instrument is (Hansen, 1985)

$$\zeta_t = \frac{1}{\sigma_m^2} E\left[\frac{\partial \mu_{t+1}}{\partial \theta} | I_t\right].$$

But the first entry of

$$E\left[\frac{\partial \mu_{t+1}}{\partial \theta} | I_t\right] = E\left[\begin{pmatrix} x_{1,t+1} x_{2,t+1}^{\alpha} \\ \beta x_{1,t+1} x_{2,t+1}^{\alpha} \log(x_{2,t+1}) \end{pmatrix} | I_t\right].$$

is β^{-1}; the second entry is

$$\begin{aligned} & \beta E\left[x_{1,t+1} x_{2,t+1}^{\alpha} \log(x_{2,t+1}) | I_t\right] = \beta E\left[\exp\left((1,\alpha) X_{t+1}\right)\left(e_2 X_{t+1}\right) | I_t\right] \\ = & \ \beta \exp\left((1,\alpha) E_X\right)\left(E\left[\exp\left((1,\alpha) U_t\right)\right] e_2 \left(E_X + \Phi\left(X_t - E_X\right)\right) + E\left[\exp\left((1,\alpha) U_t\right)\left(e_2 U_t\right)\right]\right) \\ = & \ \nu_1 \left(X_t - E_X\right) + \nu_2, \end{aligned}$$

where

$$\nu_1 = e_2 \Phi, \quad \nu_2 = e_2 \left(V_U (1,\alpha)' + E_X\right).$$

The efficiency bound equals $Q_{\zeta\partial\mu}^{-1}$, where

$$Q_{\zeta\partial\mu} \equiv \sigma_m^2 E\left[\zeta_t \zeta_t'\right] = \frac{1}{\sigma_m^2} \begin{pmatrix} \beta^{-2} & \beta^{-1}\nu_2 \\ \beta^{-1}\nu_2 & \nu_1 V_X \nu_1' + \nu_2^2 \end{pmatrix}.$$

For the two-period problem, because of conditional homoskedasticity, the optimal instrument is (Hansen, 1985)

$$\zeta_t = -\varrho_m \zeta_{t-1} + \frac{\varrho_m}{\gamma_m} \delta_t,$$

where

$$\delta_t = \sum_{i=0}^{\infty} \left(-\varrho_m\right)^i E\left[\frac{\partial \mu_{t+2+i}}{\partial \theta} | I_t\right].$$

But the first entry of

$$E\left[\frac{\partial \mu_{t+2+i}}{\partial \theta} | I_t\right] = E\left[\begin{pmatrix} 2\beta x_{1,t+1+i} x_{1,t+2+i} x_{2,t+1+i}^{\alpha} x_{2,t+2+i}^{\alpha} \\ \beta^2 x_{1,t+1+i} x_{1,t+2+i} x_{2,t+1+i}^{\alpha} x_{2,t+2+i}^{\alpha} \log(x_{2,t+1+i} x_{2,t+2+i}) \end{pmatrix} | I_t\right]$$

is $2\beta^{-1}$; the second entry is

$$\begin{aligned} & \beta^2 E\left[\exp\left((1,\alpha)\left(X_{t+1+i} + X_{t+2+i}\right)\right) e_2 \left(X_{t+1+i} + X_{t+2+i}\right) | I_t\right] \\ = & \ \beta^2 \exp\left(2(1,\alpha) E_X\right) \times \begin{pmatrix} e_2 \left(2E_X + \left(I_2 + \Phi\right)\Phi^{i+1}\left(X_t - E_X\right)\right) \times E\left[\exp\left(2(1,\alpha) U_t\right)\right] \\ + E\left[\exp\left((1,\alpha) U_t\right)\left(e_2 \left(2I_2 + \Phi\right) U_t\right)\right] \times E\left[\exp\left((1,\alpha) U_t\right)\right] \end{pmatrix} \\ = & \ e_2 \left(\left(\left(I_2 + \Phi\right)\Phi^{i+1}\left(X_t - E_X\right) + 2E_X\right)\left(1 + \gamma_m\right) + \left(2I_2 + \Phi\right) V_U (1,\alpha)'\right) \end{aligned}$$

Thus

$$\begin{aligned}\delta_t &= E\left[\sum_{i=0}^{\infty}(-\varrho_m)^i \beta x_{1,t+1+i}x_{1,t+2+i}x_{2,t+1+i}^{\alpha}x_{2,t+2+i}^{\alpha}\begin{pmatrix}2\\ \beta\log(x_{2,t+1+i}x_{2,t+2+i})\end{pmatrix}|I_t\right]\\ &= \begin{pmatrix}2\beta^{-1}(1+\varrho_m)^{-1}\\ \nu_1(X_t-E_X)+\nu_2\end{pmatrix}.\end{aligned}$$

where

$$\begin{aligned}\nu_1 &= (1+\gamma_m)e_2(I_2+\Phi)\Phi(I_2+\varrho_m\Phi)^{-1},\\ \nu_2 &= (1+\varrho_m)^{-1}e_2\left((2I_2+\Phi)V_U(1,\alpha)'+2(1+\gamma_m)E_X\right).\end{aligned}$$

The efficiency bound equals $Q_{\zeta\partial\mu}^{-1}$, where

$$Q_{\zeta\partial\mu}\equiv E\left[\zeta_t E\left[\frac{\partial\mu_{t+2}}{\partial\theta'}|I_t\right]\right]=\frac{\varrho_m}{\gamma_m}\begin{pmatrix}4\beta^{-2}(1+\varrho_m)^{-2} & 2\beta^{-1}(1+\varrho_m)^{-1}\nu_2\\ 2\beta^{-1}(1+\varrho_m)^{-1}\nu_2 & \nu_1V_X\nu_1'+\nu_2^2\end{pmatrix}.$$

5 Computation of asymptotic variance

For the one-period model, the matrix of expected outer square of the moment function is $Q_{mm}=\sigma_m^2\Gamma_0$, and the matrix of expected derivatives of the moment function is

$$Q_{\partial m}=E\left[z_t x_{1,t+1}x_{2,t+1}^{\alpha}\left(1\;\beta\log(x_{2,t+1})\right)\right]=\begin{bmatrix}\beta^{-1}\mathrm{T}_1 & \beta\mathrm{W}_1\\ \beta^{-1}\mathrm{T}_2 & \beta\mathrm{W}_2\\ \beta^{-1} & \beta W_{\circ,\circ}^{01}(1)\end{bmatrix}.$$

For the two-period model, the matrix of expected outer square of the moment function is $Q_{mm}=\sigma_m^2\Gamma_0+\gamma_m^2(\Gamma_1+\Gamma_1')$, and the matrix of expected derivatives of the moment function is

$$Q_{\partial m}=\beta E\left[z_t x_{1,t+1}x_{1,t+2}x_{2,t+1}^{\alpha}x_{2,t+2}^{\alpha}\left(2\;\beta\log(x_{2,t+1}x_{2,t+2})\right)\right]=\begin{bmatrix}2\beta^{-1}\mathrm{T}_1 & \beta^2\mathrm{A}_1\\ 2\beta^{-1}\mathrm{T}_2 & \beta^2\mathrm{A}_2\\ 2\beta^{-1} & \beta^2 A_{\circ,\circ}^{01}(1)\end{bmatrix}.$$

In these matrices,

$$\mathrm{T}_{i_1}=\left\|T_{i_1,j_1}^{0}\right\|_{j_1=0,\cdots,nl_{i_1}-1}\qquad \mathrm{W}_{i_1}=\left\|W_{i_1,j_1}^{11}(1)\right\|_{j_1=0,\cdots,nl_{i_1}-1}\qquad \mathrm{A}_{i_1}=\left\|A_{i_1,j_1}^{11}(1)\right\|_{j_1=0,\cdots,nl_{i_1}-1}$$

6 Computation of asymptotic bias

6.1 Computation of first component of $Bias_0$

The key elements of the first component of $Bias_0$ are:

$$D_{i_1,j_1,i_2,j_2}^{\pi_1\pi_2\rho_1\rho_2\rho_3\rho_4\sigma_1\sigma_2}(s)\equiv E\begin{bmatrix}x_{1,t+1-s}^{\rho_1}x_{1,t+2-s}^{\rho_2}x_{2,t+1-s}^{\rho_1\alpha}x_{2,t+2-s}^{\rho_2\alpha}x_{1,t+1}^{\rho_3}x_{1,t+2}^{\rho_4}x_{2,t+1}^{\rho_3\alpha}x_{2,t+2}^{\rho_4\alpha}\\ \times x_{i_1,t-j_1}^{\pi_1}x_{i_2,t-j_2-s}^{\pi_2}\log(x_{2,t+1}^{\sigma_1}x_{2,t+2}^{\sigma_2})\end{bmatrix}.$$

for $s\geq 0$, as well as

$$D_{i_1,j_1,i_2,-1}^{\pi_1\pi_2 00\rho_3\rho_4\sigma_1\sigma_2}(0)\equiv E\left[x_{i_1,t-j_1}^{\pi_1}x_{i_2,t+1}^{\pi_2}x_{1,t+1}^{\rho_3}x_{1,t+2}^{\rho_4}x_{2,t+1}^{\rho_3\alpha}x_{2,t+2}^{\rho_4\alpha}\log(x_{2,t+1}^{\sigma_1}x_{2,t+2}^{\sigma_2})\right].$$

For the one-period model, we need

$$\Xi\sum_{s=0}^{\infty}E\left[m_{\theta t}\Xi m_{t-s}\right].$$

Apart from the factor Ξ, the s^{th} term $E\left[m_{\theta t}\Xi m_{t-s}\right]$ is

$$\begin{bmatrix} \left\| \begin{array}{c} \sum_{i_2}\sum_{j_2}\xi^{(1)}_{i_2,j_2}\left(\beta D^{11101000}_{i_1,j_1,i_2,j_2}(s)-D^{11001000}_{i_1,j_1,i_2,j_2}(s)\right)+\beta\xi^{(2)}_{i_2,j_2}\left(\beta D^{11101010}_{i_1,j_1,i_2,j_2}(s)-D^{11001010}_{i_1,j_1,i_2,j_2}(s)\right) \\ +\xi^{(1)}\left(\beta D^{10101000}_{i_1,j_1,\circ,\circ}(s)-D^{10001000}_{i_1,j_1,\circ,\circ}(s)\right)+\beta\xi^{(2)}\left(\beta D^{10101010}_{i_1,j_1,\circ,\circ}(s)-D^{10001010}_{i_1,j_1,\circ,\circ}(s)\right) \end{array} \right\|_{i_1,j_1} \\ \sum_{i_2}\sum_{j_2}\xi^{(1)}_{i_2,j_2}\left(\beta D^{01101000}_{\circ,\circ,i_2,j_2}(s)-D^{01001000}_{\circ,\circ,i_2,j_2}(s)\right)+\beta\xi^{(2)}_{i_2,j_2}\left(\beta D^{01101010}_{\circ,\circ,i_2,j_2}(s)-D^{01001010}_{\circ,\circ,i_2,j_2}(s)\right) \\ +\xi^{(1)}\left(\beta D^{00101000}_{\circ,\circ,\circ,\circ}(s)-D^{00001000}_{\circ,\circ,\circ,\circ}(s)\right)+\beta\xi^{(2)}\left(\beta D^{00101010}_{\circ,\circ,\circ,\circ}(s)-D^{00001010}_{\circ,\circ,\circ,\circ}(s)\right) \end{bmatrix}$$

For the two-period model, we need

$$\Xi\sum_{s=-1}^{\infty}E\left[m_{\theta t}\Xi m_{t-s}\right].$$

Apart from the factor Ξ, the $(s\geq 0)^{th}$ term $E\left[m_{\theta t}\Xi m_{t-s}\right]$ is β times

$$\begin{bmatrix} \left\| \begin{array}{c} \sum_{i_2}\sum_{j_2}2\xi^{(1)}_{i_2,j_2}\left(\beta^2 D^{11111100}_{i_1,j_1,i_2,j_2}(s)-D^{11001100}_{i_1,j_1,i_2,j_2}(s)\right)+\beta\xi^{(2)}_{i_2,j_2}\left(\beta^2 D^{11111111}_{i_1,j_1,i_2,j_2}(s)-D^{11001111}_{i_1,j_1,i_2,j_2}(s)\right) \\ +2\xi^{(1)}\left(\beta^2 D^{10111100}_{i_1,j_1,\circ,\circ}(s)-D^{10001100}_{i_1,j_1,\circ,\circ}(s)\right)+\beta\xi^{(2)}\left(\beta^2 D^{10111111}_{i_1,j_1,\circ,\circ}(s)-D^{10001111}_{i_1,j_1,\circ,\circ}(s)\right) \end{array} \right\|_{i_1,j_1} \\ \sum_{i_2}\sum_{j_2}2\xi^{(1)}_{i_2,j_2}\left(\beta^2 D^{01111100}_{\circ,\circ,i_2,j_2}(s)-D^{01001100}_{\circ,\circ,i_2,j_2}(s)\right)+\beta\xi^{(2)}_{i_2,j_2}\left(\beta^2 D^{01111111}_{\circ,\circ,i_2,j_2}(s)-D^{01001111}_{\circ,\circ,i_2,j_2}(s)\right) \\ +2\xi^{(1)}\left(\beta^2 D^{00111100}_{\circ,\circ,\circ,\circ}(s)-D^{00001100}_{\circ,\circ,\circ,\circ}(s)\right)+\beta\xi^{(2)}\left(\beta^2 D^{00111111}_{\circ,\circ,\circ,\circ}(s)-D^{00001111}_{\circ,\circ,\circ,\circ}(s)\right) \end{bmatrix}$$

Apart from the factor Ξ, the $(s=-1)^{th}$ term $E\left[m_{\theta t}\Xi m_{t+1}\right]$ is β times

$$\begin{bmatrix} \left\| \begin{array}{c} \sum_{i_2}\sum_{j_2}2\xi^{(1)}_{i_2,j_2}\left(\beta D^{11001200}_{i_1,j_1,i_2,j_2-1}(0)-D^{11001100}_{i_1,j_1,i_2,j_2-1}(0)\right)+\beta\xi^{(2)}_{i_2,j_2}\left(\beta D^{11001211}_{i_1,j_1,i_2,j_2-1}(0)-D^{11001111}_{i_1,j_1,i_2,j_2-1}(0)\right) \\ +2\xi^{(1)}\left(\beta D^{10001200}_{i_1,j_1,\circ,\circ}(0)-D^{10001100}_{i_1,j_1,\circ,\circ}(0)\right)+\beta\xi^{(2)}\left(\beta D^{10001211}_{i_1,j_1,\circ,\circ}(0)-D^{10001111}_{i_1,j_1,\circ,\circ}(0)\right) \end{array} \right\|_{i_1,j_1} \\ \sum_{i_2}\sum_{j_2}2\xi^{(1)}_{i_2,j_2}\left(\beta D^{01001200}_{\circ,\circ,i_2,j_2-1}(0)-D^{01001100}_{\circ,\circ,i_2,j_2-1}(0)\right)+\beta\xi^{(2)}_{i_2,j_2}\left(\beta D^{01001211}_{\circ,\circ,i_2,j_2-1}(0)-D^{01001111}_{\circ,\circ,i_2,j_2-1}(0)\right) \\ +2\xi^{(1)}\left(\beta D^{00001200}_{\circ,\circ,\circ,\circ}(0)-D^{00001100}_{\circ,\circ,\circ,\circ}(0)\right)+\beta\xi^{(2)}\left(\beta D^{00001211}_{\circ,\circ,\circ,\circ}(0)-D^{00001111}_{\circ,\circ,\circ,\circ}(0)\right) \end{bmatrix}$$

Let us denote

$$e_\Phi(k_1,k_2)\equiv\pi_1 e_{i_1}\Phi^{k_1}+\pi_2 e_{i_2}\Phi^{k_1},\quad \Phi_\sigma\equiv\sigma_1 I_2+\sigma_2\Phi,\quad c_\sigma\equiv(\sigma_1+\sigma_2)\,e_2E_X.$$

The term $D^{\pi_1\pi_2\rho_1\rho_2\rho_3\rho_4\sigma_1\sigma_2}_{i_1,j_1,i_2,j_2}(s)$ is a product of two components. The first component is

$$\begin{array}{l} \exp\left(\left(\pi_1 e_{i_1}+\pi_2 e_{i_2}+(\rho_1+\rho_2+\rho_3+\rho_4)(1,\alpha)\right)E_X\right) \\ \times\exp\left(.5\left\{\begin{array}{ll} \rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2 & \text{if } s>1 \\ \rho_1^2+(\rho_2+\rho_3)^2+\rho_4^2 & \text{if } s=1 \\ (\rho_1+\rho_3)^2+(\rho_2+\rho_4)^2 & \text{if } s=0 \end{array}\right\}(1,\alpha)\,V_U\,(1,\alpha)'\right) \end{array}$$

Now we turn to the second component.

In the special case $\rho_1=0$, $\rho_2=0$, $j_2=-1$, $s=0$, it is

$$\begin{array}{l} \exp\left(\pi_2\rho_3 e_{i_2}V_U(1,\alpha)'+.5\pi_2^2 e_{i_2}V_\Sigma(j_1)\,e_{i_2}'\right) \\ \times\left(\begin{array}{c} \left(e_2(\rho_3\Phi_\sigma+\rho_4\sigma_2)V_U(1,\alpha)'+c_\sigma+\pi_2 e_2\Phi_\sigma V_\Sigma(j_1)\,e_{i_2}'\right) \\ \times E\left[\exp\left(e_\Phi(0,j_1+1)(X_t-E_X)\right)\right] \\ +E\left[\exp\left(e_\Phi(0,j_1+1)(X_t-E_X)\right)\left(e_2\Phi_\sigma\Phi^{j_1+1}(X_t-E_X)\right)\right] \end{array}\right). \end{array}$$

In the case $\min(j_1, j_2+s) > s-1$, it is

$$\exp\left(.5\left\{\begin{array}{ll}\pi_2^2 e_{i_2} V_\Sigma(j_1-j_2-s-1)e'_{i_2} & \text{if } j_1 \geq j_2+s \\ \pi_1^2 e_{i_1} V_\Sigma(j_2+s-j_1-1)e'_{i_1} & \text{if } j_2+s \geq j_1\end{array}\right\}\right)$$
$$\times\left(\begin{array}{c}\left(\begin{array}{c} e_2\left\{\begin{array}{ll}(\rho_1\Phi+\rho_2)\Phi_\sigma\Phi^{s-1}+\rho_3\Phi_\sigma+\rho_4\sigma_2 & \text{if } s>1 \\ (\rho_1\Phi+\rho_2+\rho_3)\Phi_\sigma+\rho_4\sigma_2 & \text{if } s=1 \\ (\rho_1+\rho_3)\Phi_\sigma+(\rho_2+\rho_4)\sigma_2 & \text{if } s=0\end{array}\right\} V_U(1,\alpha)' \\ +c_\sigma + e_2\Phi_\sigma\left\{\begin{array}{ll}\pi_2\Phi^{j_2+s+1}V_\Sigma(j_1-j_2-s-1)e'_{i_2} & \text{if } j_1\geq j_2+s \\ \pi_1\Phi^{j_1+1}V_\Sigma(j_2+s-j_1-1)e'_{i_1} & \text{if } j_2+s\geq j_1\end{array}\right\}\end{array}\right) \\ \times E\left[\exp\left(\left\{\begin{array}{ll}(e_\Phi(0,j_1-j_2-s)(X_t-E_X)) & \text{if } j_1\geq j_2+s \\ (e_\Phi(j_2+s-j_1,0)(X_t-E_X)) & \text{if } j_2+s\geq j_1\end{array}\right\}\right)\right] \\ +E\left[\exp\left(\left\{\begin{array}{ll}\begin{array}{c}(e_\Phi(0,j_1-j_2-s)(X_t-E_X)) \\ \times\left(e_2\Phi_\sigma\Phi^{j_1+1}(X_t-E_X)\right)\end{array} & \text{if } j_1\geq j_2+s \\ \begin{array}{c}(e_\Phi(j_2+s-j_1,0)(X_t-E_X)) \\ \times\left(e_2\Phi_\sigma\Phi^{j_2+s+1}(X_t-E_X)\right)\end{array} & \text{if } j_2+s\geq j_1\end{array}\right\}\right)\right]\end{array}\right).$$

In the case $j_2+s > s-1 \geq j_1$, it is

$$\exp\left(\begin{array}{c}\pi_1 e_{i_1}\left\{\begin{array}{ll}\rho_1 & \text{if } j_1=s-1 \\ \rho_1+\rho_2 & \text{if } j_1=s-2 \\ (\rho_1\Phi+\rho_2)\Phi^{s-2-j_1} & \text{if } j_1<s-2\end{array}\right\}V_U(1,\alpha)' \\ +.5\pi_1^2 e_{i_1}V_\Sigma(j_2-j_1+s-1)e'_{i_1}\end{array}\right)$$
$$\times\left(\begin{array}{c}\left(\begin{array}{c}e_2\left\{\begin{array}{ll}(\rho_1\Phi+\rho_2)\Phi_\sigma\Phi^{s-1}+\rho_3\Phi_\sigma+\rho_4\sigma_2 & \text{if } s>1 \\ (\rho_1\Phi+\rho_2+\rho_3)\Phi_\sigma+\rho_4\sigma_2 & \text{if } s=1\end{array}\right\}V_U(1,\alpha)' \\ +c_\sigma+\pi_1 e_2\Phi_\sigma\Phi^{j_1+1}V_\Sigma(j_2+s-j_1-1)e'_{i_1}\end{array}\right) \\ \times E\left[\exp\left(e_\Phi(j_2+s-j_1,0)(X_t-E_X)\right)\right] \\ +E\left[\exp\left(e_\Phi(j_2+s-j_1,0)(X_t-E_X)\right)\left(e_2\Phi_\sigma\Phi^{j_2+s+1}(X_t-E_X)\right)\right]\end{array}\right).$$

6.2 Computation of second component of $Bias_0$

For the one-period problem, the second derivatives of the moment function are

$$\begin{aligned}\frac{\partial m_{\theta t}}{\partial\beta} &= z_t x_{1,t+1}x_{2,t+1}^\alpha\left(0\ \log(x_{2,t+1})\right), \\ \frac{\partial m_{\theta t}}{\partial\alpha} &= z_t x_{1,t+1}x_{2,t+1}^\alpha \log(x_{2,t+1})\left(1\ \beta\log(x_{2,t+1})\right),\end{aligned}$$

so the second component of $Bias_0$, apart from the factor $-\Xi$, is

$$\left[\begin{array}{c}\left\|\Sigma_{12}W^{11}_{i_1,j_1}(1)+.5\beta\Sigma_{22}W^{11}_{i_1,j_1}(2)\right\|_{i_1,j_1} \\ \Sigma_{12}W^{01}_{\circ,\circ}(1)+.5\beta\Sigma_{22}W^{01}_{\circ,\circ}(2)\end{array}\right].$$

For the two-period problem, the second derivatives of the moment function are

$$\begin{aligned}\frac{\partial m_{\theta t}}{\partial\beta} &= 2z_t x_{1,t+1}x_{1,t+2}x_{2,t+1}^\alpha x_{2,t+2}^\alpha\left(1\ \beta\log(x_{2,t+1}x_{2,t+2})\right), \\ \frac{\partial m_{\theta t}}{\partial\alpha} &= \beta z_t x_{1,t+1}x_{1,t+2}x_{2,t+1}^\alpha x_{2,t+2}^\alpha \log(x_{2,t+1}x_{2,t+2})\left(2\ \beta\log(x_{2,t+1}x_{2,t+2})\right).\end{aligned}$$

The second component of $Bias_0$, apart from the factor $-\Xi$, is

$$\left[\begin{array}{c}\left\|\beta^{-2}\Sigma_{11}T^0_{i_1,j_1}+2\beta\Sigma_{12}A^{11}_{i_1,j_1}(1)+.5\beta^2\Sigma_{22}A^{11}_{i_1,j_1}(2)\right\|_{i_1,j_1} \\ \beta^{-2}\Sigma_{11}+2\beta\Sigma_{12}A^{01}_{\circ,\circ}(1)+.5\beta^2\Sigma_{22}A^{01}_{\circ,\circ}(2)\end{array}\right].$$

6.3 Computation of $Bias_1$

For the one-period model, we need

$$Bias_1(s) = -\Sigma E\left[m'_{\theta t}\Omega m_{t-s}\right],$$

where s may vary from 0 to ∞. Apart from the factor $-\Sigma$, it is

$$\sum_{i_1,i_2}\sum_{j_1,j_2}\omega_{i_1,j_1,i_2,j_2}\begin{pmatrix}\beta D^{11101000}_{i_1,j_1,i_2,j_2}(s) - D^{11001000}_{i_1,j_1,i_2,j_2}(s)\\ \beta^2 D^{11101010}_{i_1,j_1,i_2,j_2}(s) - \beta D^{11001010}_{i_1,j_1,i_2,j_2}(s)\end{pmatrix} + \omega\begin{pmatrix}\beta D^{00101000}_{\circ,\circ,\circ,\circ}(s) - D^{00001000}_{\circ,\circ,\circ,\circ}(s)\\ \beta^2 D^{00101010}_{\circ,\circ,\circ,\circ}(s) - \beta D^{00001010}_{\circ,\circ,\circ,\circ}(s)\end{pmatrix}$$
$$+\sum_{i_1}\sum_{j_1}\omega_{i_1,j_1}\begin{pmatrix}\beta D^{10101000}_{i_1,j_1,\circ,\circ}(s) - D^{10001000}_{i_1,j_1,\circ,\circ}(s)\\ \beta^2 D^{10101010}_{i_1,j_1,\circ,\circ}(s) - \beta D^{10001010}_{i_1,j_1,\circ,\circ}(s)\end{pmatrix} + \sum_{i_2}\sum_{j_2}\omega_{i_2,j_2}\begin{pmatrix}\beta D^{01101000}_{\circ,\circ,i_2,j_2}(s) - D^{01001000}_{\circ,\circ,i_2,j_2}(s)\\ \beta^2 D^{01101010}_{\circ,\circ,i_2,j_2}(s) - \beta D^{01001010}_{\circ,\circ,i_2,j_2}(s)\end{pmatrix}$$

For the two-period model, we need

$$Bias_1(s) = -\Sigma E\left[m'_{\theta t}\Omega m_{t-s}\right],$$

where s may vary from -1 to ∞. Apart from the factor $-\Sigma\beta$, for $s \geq 0$ it is

$$\sum_{i_1,i_2}\sum_{j_1,j_2}\omega_{i_1,j_1,i_2,j_2}\begin{pmatrix}2\beta^2 D^{11111100}_{i_1,j_1,i_2,j_2}(s) - 2D^{11001100}_{i_1,j_1,i_2,j_2}(s)\\ \beta^3 D^{11111111}_{i_1,j_1,i_2,j_2}(s) - \beta D^{11001111}_{i_1,j_1,i_2,j_2}(s)\end{pmatrix} + \omega\begin{pmatrix}2\beta^2 D^{00111100}_{\circ,\circ,\circ,\circ}(s) - 2D^{00001100}_{\circ,\circ,\circ,\circ}(s)\\ \beta^3 D^{00111111}_{\circ,\circ,\circ,\circ}(s) - \beta D^{00001111}_{\circ,\circ,\circ,\circ}(s)\end{pmatrix}$$
$$+\sum_{i_1}\sum_{j_1}\omega_{i_1,j_1}\begin{pmatrix}2\beta^2 D^{10111100}_{i_1,j_1,\circ,\circ}(s) - 2D^{10001100}_{i_1,j_1,\circ,\circ}(s)\\ \beta^3 D^{10111111}_{i_1,j_1,\circ,\circ}(s) - \beta D^{10001111}_{i_1,j_1,\circ,\circ}(s)\end{pmatrix} + \sum_{i_2}\sum_{j_2}\omega_{i_2,j_2}\begin{pmatrix}2\beta^2 D^{01111100}_{\circ,\circ,i_2,j_2}(s) - 2D^{01001100}_{\circ,\circ,i_2,j_2}(s)\\ \beta^3 D^{01111111}_{\circ,\circ,i_2,j_2}(s) - \beta D^{01001111}_{\circ,\circ,i_2,j_2}(s)\end{pmatrix}$$

Apart from the factor $-\Sigma\beta$, for $s = -1$ it is $E\left[m'_{\theta t}\Omega m_{t+1}\right]$, or

$$\sum_{i_1,i_2}\sum_{j_1,j_2}\omega_{i_1,j_1,i_2,j_2}\begin{pmatrix}2\beta D^{11001200}_{i_1,j_1,i_2,j_2-1}(0) - 2D^{11001100}_{i_1,j_1,i_2,j_2-1}(0)\\ \beta^2 D^{11001211}_{i_1,j_1,i_2,j_2-1}(0) - \beta D^{11001111}_{i_1,j_1,i_2,j_2-1}(0)\end{pmatrix} + \omega\begin{pmatrix}2\beta D^{00001200}_{\circ,\circ,\circ,\circ}(0) - 2D^{00001100}_{\circ,\circ,\circ,\circ}(0)\\ \beta^2 D^{00001211}_{\circ,\circ,\circ,\circ}(0) - \beta D^{00001111}_{\circ,\circ,\circ,\circ}(0)\end{pmatrix}$$
$$+\sum_{i_1}\sum_{j_1}\omega_{i_1,j_1}\begin{pmatrix}2\beta D^{10001200}_{i_1,j_1,\circ,\circ}(0) - 2D^{10001100}_{i_1,j_1,\circ,\circ}(0)\\ \beta^2 D^{10001211}_{i_1,j_1,\circ,\circ}(0) - \beta D^{10001111}_{i_1,j_1,\circ,\circ}(0)\end{pmatrix} + \sum_{i_2}\sum_{j_2}\omega_{i_2,j_2}\begin{pmatrix}2\beta D^{01001200}_{\circ,\circ,i_2,j_2-1}(0) - 2D^{01001100}_{\circ,\circ,i_2,j_2-1}(0)\\ \beta^2 D^{01001211}_{\circ,\circ,i_2,j_2-1}(0) - \beta D^{01001111}_{\circ,\circ,i_2,j_2-1}(0)\end{pmatrix}$$

6.4 Computation of $Bias_2$

The key elements of $Bias_2$ are:

$$F^{\pi_1\pi_2\pi_3\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) \equiv E\begin{bmatrix} x^{\rho_1}_{1,t+1-r}x^{\rho_2}_{1,t+2-r}x^{\rho_1\alpha}_{2,t+1-r}x^{\rho_2\alpha}_{2,t+2-r}x^{\rho_3}_{1,t+1-s}x^{\rho_4}_{1,t+2-s}x^{\rho_3\alpha}_{2,t+1-s}x^{\rho_4\alpha}_{2,t+2-s}\\ \times x^{\pi_1}_{i_1,t-j_1}x^{\pi_2}_{i_2,t-j_2-r}x^{\pi_3}_{i_3,t-j_3-s}x^{\rho_5}_{1,t+1}x^{\rho_6}_{1,t+2}x^{\rho_5\alpha}_{2,t+1}x^{\rho_6\alpha}_{2,t+2}\end{bmatrix}$$

with s running from -2 to ∞, and $r = 0, \pm 1$.

For the one-period model, we need

$$Bias_2(s) = \Xi E\left[m_t m'_t\Omega m_{t-s}\right],$$

with s running from 0 to ∞. Apart from the factor Ξ, the s^{th} term is

$$\begin{aligned} E\left[m_t m'_t\Omega m_{t-s}\right] &= E\left[z_t\left(z'_t\Omega z_{t-s}\right)\left(\beta x_{1,t+1}x^{\alpha}_{2,t+1}-1\right)^2\left(\beta x_{1,t+1-s}x^{\alpha}_{2,t+1-s}-1\right)\right] \\ &= \begin{bmatrix} \left\| \begin{matrix} \sum\limits_{i_2,i_3}\sum\limits_{j_2,j_3}\omega_{i_2,j_2,i_3,j_3}F^{111}_{i_1,j_1,i_2,j_2,i_3,j_3}(s) + \sum\limits_{i_2}\sum\limits_{j_2}\omega_{i_2,j_2}F^{110}_{i_1,j_1,i_2,j_2,\circ,\circ}(s) \\ +\sum\limits_{i_3}\sum\limits_{j_3}\omega_{i_3,j_3}F^{101}_{i_1,j_1,\circ,\circ,i_3,j_3}(s) + \omega F^{100}_{i_1,j_1,\circ,\circ,\circ,\circ}(s)\end{matrix}\right\|_{i_1,j_1} \\ \sum\limits_{i_2,i_3}\sum\limits_{j_2,j_3}\omega_{i_2,j_2,i_3,j_3}F^{011}_{\circ,\circ,i_2,j_2,i_3,j_3}(s) + \sum\limits_{i_2}\sum\limits_{j_2}\omega_{i_2,j_2}F^{010}_{\circ,\circ,i_2,j_2,\circ,\circ}(s) \\ +\sum\limits_{i_3}\sum\limits_{j_3}\omega_{i_3,j_3}F^{001}_{\circ,\circ,\circ,\circ,i_3,j_3}(s) + \omega F^{000}_{\circ,\circ,\circ,\circ,\circ,\circ}(s)\end{bmatrix} \end{aligned}$$

where

$$F^{\pi_1\pi_2\pi_3}_{i_1,j_1,i_2,j_2,i_3,j_3}(s) = \beta^3 F^{\pi_1\pi_2\pi_3 001020}_{i_1,j_1,i_2,j_2,i_3,j_3}(\circ,s) - F^{\pi_1\pi_2\pi_3 000000}_{i_1,j_1,i_2,j_2,i_3,j_3}(\circ,s) \\ -\beta^2\left(2F^{\pi_1\pi_2\pi_3 001010}_{i_1,j_1,i_2,j_2,i_3,j_3}(\circ,s) + F^{\pi_1\pi_2\pi_3 000020}_{i_1,j_1,i_2,j_2,i_3,j_3}(\circ,s)\right) \\ +\beta\left(2F^{\pi_1\pi_2\pi_3 000010}_{i_1,j_1,i_2,j_2,i_3,j_3}(\circ,s) + F^{\pi_1\pi_2\pi_3 001000}_{i_1,j_1,i_2,j_2,i_3,j_3}(\circ,s)\right)$$

For the two-period model, we need

$$Bias_2(s) = \Xi E\left[m_t\left(m_t + m_{t-1} + m_{t+1}\right)' \Omega m_{t-s}\right],$$

with s running from -2 to ∞. Apart from the factor Ξ, the s^{th} term consists of three components, for $r = 0, \pm 1$, of the type

$$E\left[m_t m'_{t-r}\Omega m_{t-s}\right] = E\begin{bmatrix} z_t\left(z'_{t-r}\Omega z_{t-s}\right)\left(\beta^2 x_{1,t+1}x_{1,t+2}x^{\alpha}_{2,t+1}x^{\alpha}_{2,t+2} - 1\right) \\ \times\left(\beta^2 x_{1,t+1-r}x_{1,t+2-r}x^{\alpha}_{2,t+1-r}x^{\alpha}_{2,t+2-r} - 1\right) \\ \times\left(\beta^2 x_{1,t+1-s}x_{1,t+2-s}x^{\alpha}_{2,t+1-s}x^{\alpha}_{2,t+2-s} - 1\right)\end{bmatrix}$$

$$= \begin{bmatrix} \left\| \begin{array}{l} \sum_{i_2,i_3}\sum_{j_2,j_3}\omega_{i_2,j_2,i_3,j_3}F^{111}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) + \sum_{i_2}\sum_{j_2}\omega_{i_2,j_2}F^{110}_{i_1,j_1,i_2,j_2,\circ,\circ}(r,s) \\ +\sum_{i_3}\sum_{j_3}\omega_{i_3,j_3}F^{101}_{i_1,j_1,\circ,\circ,i_3,j_3}(r,s) + \omega F^{100}_{i_1,j_1,\circ,\circ,\circ,\circ}(r,s) \end{array}\right\|_{i_1,j_1} \\ \sum_{i_2,i_3}\sum_{j_2,j_3}\omega_{i_2,j_2,i_3,j_3}F^{011}_{\circ,\circ,i_2,j_2,i_3,j_3}(r,s) + \sum_{i_2}\sum_{j_2}\omega_{i_2,j_2}F^{010}_{\circ,\circ,i_2,j_2,\circ,\circ}(r,s) \\ +\sum_{i_3}\sum_{j_3}\omega_{i_3,j_3}F^{001}_{\circ,\circ,\circ,\circ,i_3,j_3}(r,s) + \omega F^{000}_{\circ,\circ,\circ,\circ,\circ,\circ}(r,s)\end{bmatrix}$$

where

$$F^{\pi_1\pi_2\pi_3}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) = \beta^6 F^{\pi_1\pi_2\pi_3 111111}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) - F^{\pi_1\pi_2\pi_3 000000}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) \\ -\beta^4\left(F^{\pi_1\pi_2\pi_3 001111}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) + F^{\pi_1\pi_2\pi_3 110011}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) + F^{\pi_1\pi_2\pi_3 111100}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s)\right) \\ +\beta^2\left(F^{\pi_1\pi_2\pi_3 110000}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) + F^{\pi_1\pi_2\pi_3 001100}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s) + F^{\pi_1\pi_2\pi_3 000011}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s)\right)$$

Let us denote

$$e_\Phi(k_1,k_2,k_3) = \pi_1 e_{i_1}\Phi^{k_1} + \pi_2 e_{i_2}\Phi^{k_2} + \pi_3 e_{i_3}\Phi^{k_3}$$

$$V_{\alpha\pi}(\rho,k_1,k_2,k_3) = \left(\rho(1,\alpha) + \pi_1 e_{i_1}\Phi^{k_1} + \pi_2 e_{i_2}\Phi^{k_2} + \pi_3 e_{i_3}\Phi^{k_3}\right) V_U \left(\rho(1,\alpha) + \pi_1 e_{i_1}\Phi^{k_1} + \pi_2 e_{i_2}\Phi^{k_2} + \pi_3 e_{i_3}\Phi^{k_3}\right)'$$

$$\sigma^2_{r,s} = \exp\left(.5\left\{\begin{array}{ll} (\rho_1+\rho_6)^2 + (\rho_2+\rho_3)^2 + \rho_4^2 + \rho_5^2 & \text{if } r=-1, s=-2 \\ (\rho_2+\rho_4)^2 + (\rho_1+\rho_3+\rho_6)^2 + \rho_5^2 & \text{if } r=-1, s=-1 \\ (\rho_1+\rho_4+\rho_6)^2 + \rho_2^2 + (\rho_3+\rho_5)^2 & \text{if } r=-1, s=0 \\ (\rho_1+\rho_6)^2 + \rho_2^2 + \rho_3^2 + (\rho_4+\rho_5)^2 & \text{if } r=-1, s=+1 \\ (\rho_1+\rho_6)^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 + \rho_5^2 & \text{if } r=-1, \text{ other } s \\ (\rho_1+\rho_5)^2 + (\rho_2+\rho_3+\rho_6)^2 + \rho_4^2 & \text{if } r=0, s=-1 \\ (\rho_1+\rho_3+\rho_5)^2 + (\rho_2+\rho_4+\rho_6)^2 & \text{if } r=0, s=0 \\ (\rho_1+\rho_4+\rho_5)^2 + (\rho_2+\rho_6)^2 + \rho_3^2 & \text{if } r=0, s=+1 \\ (\rho_1+\rho_5)^2 + (\rho_2+\rho_6)^2 + \rho_3^2 + \rho_4^2 & \text{if } r=0, |s|>1 \\ (\rho_2+\rho_5)^2 + (\rho_3+\rho_6)^2 + \rho_1^2 + \rho_4^2 & \text{if } r=+1, s=-1 \\ (\rho_2+\rho_3+\rho_5)^2 + (\rho_4+\rho_6)^2 + \rho_1^2 & \text{if } r=+1, s=0 \\ (\rho_1+\rho_3)^2 + (\rho_2+\rho_4+\rho_5)^2 + \rho_6^2 & \text{if } r=+1, s=+1 \\ (\rho_1+\rho_4)^2 + (\rho_2+\rho_5)^2 + \rho_3^2 + \rho_6^2 & \text{if } r=+1, s=+2 \\ \rho_1^2 + (\rho_2+\rho_5)^2 + \rho_3^2 + \rho_4^2 + \rho_6^2 & \text{if } r=+1, \text{ other } s \end{array}\right\}(1,\alpha)V_U(1,\alpha)'\right)$$

$$\sigma^2_{s>r} = \exp\left(.5\left\{\begin{array}{ll} (\rho_1+\rho_4+\rho_6)^2 + \rho_2^2 + \rho_5^2 & \text{if } r=-1, s=0 \\ (\rho_1+\rho_6)^2 + (\rho_4+\rho_5)^2 + \rho_2^2 & \text{if } r=-1, s=1 \\ (\rho_1+\rho_6)^2 + \rho_2^2 + \rho_4^2 + \rho_5^2 & \text{if } r=-1, s>1 \\ (\rho_1+\rho_4+\rho_5)^2 + (\rho_2+\rho_6)^2 & \text{if } r=0, s=1 \\ (\rho_1+\rho_5)^2 + (\rho_2+\rho_6)^2 + \rho_4^2 & \text{if } r=0, s>1 \\ (\rho_1+\rho_4)^2 + (\rho_2+\rho_5)^2 + \rho_6^2 & \text{if } r=+1, s=2 \\ \rho_1^2 + (\rho_2+\rho_5)^2 + \rho_4^2 + \rho_6^2 & \text{if } r=+1, s>2 \end{array}\right\}(1,\alpha)V_U(1,\alpha)'\right)$$

$$\sigma^2_{r>s} = \exp\left(.5 \left\{ \begin{array}{ll} (\rho_2+\rho_3)^2+\rho_4^2+\rho_5^2+\rho_6^2 & \text{if } r=-1, s=-2 \\ (\rho_2+\rho_6)^2+\rho_3^2+\rho_4^2+\rho_5^2 & \text{if } r=0, s=-2 \\ (\rho_2+\rho_3+\rho_6)^2+\rho_4^2+\rho_5^2 & \text{if } r=0, s=-1 \\ (\rho_2+\rho_5)^2+\rho_3^2+\rho_4^2+\rho_6^2 & \text{if } r=+1, s=-2 \\ (\rho_2+\rho_5)^2+(\rho_3+\rho_6)^2+\rho_4^2 & \text{if } r=+1, s=-1 \\ (\rho_2+\rho_3+\rho_5)^2+(\rho_4+\rho_6)^2 & \text{if } r=+1, s=0 \end{array} \right\} (1,\alpha) V_U (1,\alpha)' \right)$$

$$\sigma^2_{r} = \exp\left(.5 \left\{ \begin{array}{ll} (\rho_1+\rho_6)^2+\rho_2^2+\rho_5^2 & \text{if } r=-1 \\ (\rho_1+\rho_5)^2+(\rho_2+\rho_6)^2 & \text{if } r=0 \\ \rho_1^2+(\rho_2+\rho_5)^2+\rho_6^2 & \text{if } r=+1 \end{array} \right\} (1,\alpha) V_U (1,\alpha)' \right)$$

$$\sigma^2_{s} = \exp\left(.5 \left\{ \begin{array}{ll} \rho_4^2+\rho_5^2+(\rho_3+\rho_6)^2 & \text{if } s=-1 \\ (\rho_3+\rho_5)^2+(\rho_4+\rho_6)^2 & \text{if } s=0 \\ \rho_3^2+(\rho_4+\rho_5)^2+\rho_6^2 & \text{if } s=+1 \\ \rho_3^2+\rho_4^2+\rho_5^2+\rho_6^2 & \text{if } |s|>1 \end{array} \right\} (1,\alpha) V_U (1,\alpha)' \right)$$

$$\sigma^2_{s=r=1} = \exp\left(.5 \left((\rho_2+\rho_4+\rho_5)^2+\rho_6^2 \right) (1,\alpha) V_U (1,\alpha)' \right)$$

$$\sigma^2_{r=1} = \exp\left(.5 \left((\rho_2+\rho_5)^2+\rho_6^2 \right) (1,\alpha) V_U (1,\alpha)' \right)$$

$$\sigma^2_{-} = \exp\left(.5 \left(\rho_5^2+\rho_6^2 \right) (1,\alpha) V_U (1,\alpha)' \right)$$

The term $F^{\pi_1\pi_2\pi_3\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}_{i_1,j_1,i_2,j_2,i_3,j_3}(r,s)$ is a product of two components. The first component is

$$\begin{array}{l} \exp\left((\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6)(1,\alpha) E_X + (\pi_1 e_{i_1}+\pi_2 e_{i_2}+\pi_3 e_{i_3}) E_X \right) \\ \times E\left[\exp \left\{ \begin{array}{ll} e_\Phi (0, j_1-j_2-r, j_1-j_3-s)(X_t-E_X) & \text{if } j_1 \geq \max(j_2+r, j_3+s) \\ e_\Phi (j_2+r-j_1, 0, j_2+r-j_3-s)(X_t-E_X) & \text{if } j_2+r \geq \max(j_1, j_3+s) \\ e_\Phi (j_3+s-j_1, j_3+s-j_2-r, 0)(X_t-E_X) & \text{if } j_3+s \geq \max(j_1, j_2+r) \end{array} \right\} \right] \end{array}$$

Now we turn to the second component. In the case $\min(j_1, j_2+r, j_3+s) > \max(r-1, s-1)$, it is

$$\sigma^2_{r,s} \exp\left(.5 \left\{ \begin{array}{ll} \left(\begin{array}{l} \pi_3^2 e_{i_3} V_\Sigma (j_2+r-j_3-s-1) e'_{i_3} + \\ \left(\pi_2 e_{i_2}+\pi_3 e_{i_3}\Phi^{j_2+r-j_3-s}\right) V_\Sigma (j_1-j_2-r-1) \\ \quad \times \left(\pi_2 e_{i_2}+\pi_3 e_{i_3}\Phi^{j_2-j_3-s+r}\right)' \end{array} \right) & \text{if } j_1 \geq j_2+r \geq j_3+s \\ \left(\begin{array}{l} \pi_2^2 e_{i_2} V_\Sigma (j_3+s-j_2-r-1) e'_{i_2} + \\ \left(\pi_2 e_{i_2}\Phi^{j_3+s-j_2-r}+\pi_3 e_{i_3}\right) V_\Sigma (j_1-j_3-s-1) \\ \quad \times \left(\pi_2 e_{i_2}\Phi^{j_3+s-j_2-r}+\pi_3 e_{i_3}\right)' \end{array} \right) & \text{if } j_1 \geq j_3+s \geq j_2+r \\ \left(\begin{array}{l} \pi_3^2 e_{i_3} V_\Sigma (j_1-j_3-s-1) e'_{i_3} + \\ \left(\pi_1 e_{i_1}+\pi_3 e_{i_3}\Phi^{j_1-j_3-s}\right) V_\Sigma (j_2+r-j_1-1) \\ \quad \times \left(\pi_1 e_{i_1}+\pi_3 e_{i_3}\Phi^{j_1-j_3-s}\right)' \end{array} \right) & \text{if } j_2+r \geq j_1 \geq j_3+s \\ \left(\begin{array}{l} \pi_1^2 e_{i_1} V_\Sigma (j_3+s-j_1-1) e'_{i_1} + \\ \left(\pi_1 e_{i_1}\Phi^{j_3+s-j_1}+\pi_3 e_{i_3}\right) V_\Sigma (j_2+r-j_3-s-1) \\ \quad \times \left(\pi_1 e_{i_1}\Phi^{j_3+s-j_1}+\pi_3 e_{i_3}\right)' \end{array} \right) & \text{if } j_2+r \geq j_3+s \geq j_1 \\ \left(\begin{array}{l} \pi_2^2 e_{i_2} V_\Sigma (j_1-j_2-r-1) e'_{i_2} + \\ \left(\pi_1 e_{i_1}+\pi_2 e_{i_2}\Phi^{j_1-j_2-r}\right) V_\Sigma (j_3+s-j_1-1) \\ \quad \times \left(\pi_1 e_{i_1}+\pi_2 e_{i_2}\Phi^{j_1-j_2-r}\right)' \end{array} \right) & \text{if } j_3+s \geq j_1 \geq j_2+r \\ \left(\begin{array}{l} \pi_1^2 e_{i_1} V_\Sigma (j_2+r-j_1-1) e'_{i_1} + \\ \left(\pi_1 e_{i_1}\Phi^{j_2+r-j_1}+\pi_2 e_{i_2}\right) V_\Sigma (j_3+s-j_2-r-1) \\ \quad \times \left(\pi_1 e_{i_1}\Phi^{j_2+r-j_1}+\pi_2 e_{i_2}\right)' \end{array} \right) & \text{if } j_3+s \geq j_2+r \geq j_1 \end{array} \right\} \right).$$

In the case $j_1 \geq j_2+r > j_3+s = r-1 > s-1$, it is

$$\sigma^2_{r>s} \exp\left(\begin{array}{c} .5 V_{\alpha\pi}(\rho_1,-,-,0) + .5\pi_3^2 e_{i_3} \Phi V_\Sigma (j_2+r-j_3-s-2) \Phi' e'_{i_3} \\ +.5\left(\pi_2 e_{i_2}+\pi_3 e_{i_3}\Phi^{j_2+r-j_3-s}\right) V_\Sigma (j_1-j_2-r-1) \left(\pi_2 e_{i_2}+\pi_3 e_{i_3}\Phi^{j_2+r-j_3-s}\right)' \end{array} \right)$$

In the case $j_1 \geq j_3 + s > j_2 + r = s - 1 > r - 1$, it is

$$\sigma^2_{s>r} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_3, -, 0, -) + .5\pi_2^2 e_{i_2} \Phi V_\Sigma (j_3 + s - j_2 - r - 2) \Phi' e'_{i_2} \\ +.5 \left(\pi_2 e_{i_2} \Phi^{j_3+s-j_2-r} + \pi_3 e_{i_3}\right) V_\Sigma (j_1 - j_3 - s - 1) \left(\pi_2 e_{i_2} \Phi^{j_3+s-j_2-r} + \pi_3 e_{i_3}\right)' \end{pmatrix}$$

In the case $j_1 \geq j_2 + r > r - 1 > j_3 + s > s - 1$, it is

$$\sigma^2_{s} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_1, -, -, r - j_3 - s - 1) + .5V_{\alpha\pi}(\rho_2, -, -, r - j_3 - s - 2) \\ +.5\pi_3^2 e_{i_3} V_\Sigma (r - j_3 - s - 3) e'_{i_3} + .5\pi_3^2 e_{i_3} \Phi^{r-j_3-s} V_\Sigma (j_2 - 1) \Phi^{r-j_3-s\prime} e'_{i_3} \\ +.5 \left(\pi_2 e_{i_2} + \pi_3 e_{i_3} \Phi^{j_2+r-j_3-s}\right) V_\Sigma (j_1 - j_2 - r - 1) \left(\pi_2 e_{i_2} + \pi_3 e_{i_3} \Phi^{j_2+r-j_3-s}\right)' \end{pmatrix}$$

In the case $j_1 \geq j_3 + s > s - 1 > j_2 + r > r - 1$, it is

$$\sigma^2_{r} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_3, -, s - j_2 - r - 1, -) + .5V_{\alpha\pi}(\rho_4, -, s - j_2 - r - 2, -) \\ +.5\pi_2^2 e_{i_2} V_\Sigma (s - j_2 - r - 3) e'_{i_2} + .5\pi_2^2 e_{i_2} \Phi^{s-j_2-r} V_\Sigma (j_3 - 1) \Phi^{s-j_2-r\prime} e'_{i_2} \\ +.5 \left(\pi_2 e_{i_2} \Phi^{j_3+s-j_2-r} + \pi_3 e_{i_3}\right) V_\Sigma (j_1 - j_3 - s - 1) \left(\pi_2 e_{i_2} \Phi^{j_3+s-j_2-r} + \pi_3 e_{i_3}\right)' \end{pmatrix}$$

In the case $j_2 + r > j_1 \geq j_3 + s = r - 1 > s - 1$, it is

$$\sigma^2_{r>s} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_1, -, -, 0) + .5\pi_3^2 e_{i_3} \Phi V_\Sigma (j_1 - j_3 - s - 2) \Phi' e'_{i_3} \\ +.5 \left(\pi_1 e_{i_1} + (\rho_1(1,\alpha) + \pi_3 e_{i_3}) \Phi^{j_1-j_3-s}\right) V_\Sigma (j_2 + r - j_1 - 1) \\ \times \left(\pi_1 e_{i_1} + (\rho_1(1,\alpha) + \pi_3 e_{i_3}) \Phi^{j_1-j_3-s}\right)' \end{pmatrix}$$

In the case $j_3 + s > j_1 \geq j_2 + r = s - 1 > r - 1$, it is

$$\sigma^2_{s>r} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_3, -, 0, -) + .5\pi_2^2 e_{i_2} \Phi V_\Sigma (j_1 - j_2 - r - 2) \Phi' e'_{i_2} \\ +.5 \left(\pi_1 e_{i_1} + (\rho_3(1,\alpha) + \pi_2 e_{i_2}) \Phi^{j_1-j_2-r}\right) V_\Sigma (j_3 + s - j_1 - 1) \\ \times \left(\pi_1 e_{i_1} + (\rho_3(1,\alpha) + \pi_2 e_{i_2}) \Phi^{j_1-j_2-r}\right)' \end{pmatrix}$$

In the case $j_2 + r > j_1 \geq r - 1 > j_3 + s > s - 1$, it is

$$\sigma^2_{s} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_1, -, -, r - j_3 - s - 1) + .5V_{\alpha\pi}(\rho_2, -, -, r - j_3 - s - 2) \\ +.5\pi_3^2 e_{i_3} V_\Sigma (r - j_3 - s - 3) e'_{i_3} + .5\pi_3^2 e_{i_3} \Phi^{r-j_3-s} V_\Sigma (j_1 - r - 1) \Phi^{r-j_3-s\prime} e'_{i_3} \end{pmatrix}$$

In the case $j_3 + s > j_1 \geq s - 1 > j_2 + r > r - 1$, it is

$$\sigma^2_{r} \exp \begin{pmatrix} .5V_{\alpha\pi}(\rho_3, -, s - j_2 - r - 1, -) + .5V_{\alpha\pi}(\rho_4, -, s - j_2 - r - 2, -) \\ +.5\pi_2^2 e_{i_2} V_\Sigma (s - j_2 - r - 3) e'_{i_2} + .5\pi_2^2 e_{i_2} \Phi^{s-j_2-r} V_\Sigma (j_1 - s - 1) \Phi^{s-j_2-r\prime} e'_{i_2} \end{pmatrix}$$

In the case $\min(j_2 + r, j_3 + s) > j_1 = \begin{cases} r - 1 \geq s - 1 \\ s - 1 > r - 1 \end{cases}$, it is

$$\begin{Bmatrix} \sigma^2_{r>s} \exp(.5V_{\alpha\pi}(\rho_1, 0, -, -)) & \text{if } j_1 = r - 1 > s - 1 \\ \sigma^2_{s=r=1} \exp(.5V_{\alpha\pi}(\rho_1 + \rho_3, 0, -, -)) & \text{if } j_1 = r - 1 = s - 1 \\ \sigma^2_{s>r} \exp(.5V_{\alpha\pi}(\rho_3, 0, -, -)) & \text{if } j_1 = s - 1 > r - 1 \end{Bmatrix}$$
$$\times \exp \begin{Bmatrix} \begin{array}{c} .5\pi_1^2 e_{i_1} \Phi V_\Sigma (j_3 + s - j_1 - 2) \Phi' e'_{i_1} + .5 \left(\pi_1 e_{i_1} \Phi^{j_3+s-j_1} + \pi_3 e_{i_3}\right) \\ \times V_\Sigma (j_2 + r - j_3 - s - 1) \left(\pi_1 e_{i_1} \Phi^{j_3+s-j_1} + \pi_3 e_{i_3}\right)' \end{array} & \text{if } j_2 + r \geq j_3 + s \\ \begin{array}{c} .5\pi_1^2 e_{i_1} \Phi V_\Sigma (j_2 + r - j_1 - 2) \Phi' e'_{i_1} + .5 \left(\pi_1 e_{i_1} \Phi^{j_2+r-j_1} + \pi_2 e_{i_2}\right) \\ \times V_\Sigma (j_3 + s - j_2 - r - 1) \left(\pi_1 e_{i_1} \Phi^{j_2+r-j_1} + \pi_2 e_{i_2}\right)' \end{array} & \text{if } j_3 + s \geq j_2 + r \end{Bmatrix}$$

In the case $\min(j_3 + s, j_2 + r) > s - 1 > j_1 \geq r - 1$, it is

$$\begin{Bmatrix} \sigma^2_{r} \exp\left(.5\pi_1^2 e_{i_1} V_\Sigma (s - j_1 - 3) e'_{i_1}\right) & \text{if } j_1 > r - 1 \\ \sigma^2_{r=1} \exp\left(.5V_{\alpha\pi}(\rho_1, 0, -, -) + .5\pi_1^2 e_{i_1} \Phi V_\Sigma (s - j_1 - 4) \Phi' e'_{i_1}\right) & \text{if } j_1 = r - 1 \end{Bmatrix}$$
$$\times \exp(.5V_{\alpha\pi}(\rho_3, s - j_1 - 1, -, -) + .5V_{\alpha\pi}(\rho_4, s - j_1 - 2, -, -))$$
$$\times \exp \begin{Bmatrix} \begin{array}{c} .5\pi_1^2 e_{i_1} \Phi^{s-j_1} V_\Sigma (j_3 - 1) \Phi^{s-j_1\prime} e'_{i_1} + .5 \left(\pi_1 e_{i_1} \Phi^{j_3+s-j_1} + \pi_3 e_{i_3}\right) \\ \times V_\Sigma (j_2 + r - j_3 - s - 1) \left(\pi_1 e_{i_1} \Phi^{j_3+s-j_1} + \pi_3 e_{i_3}\right)' \end{array} & \text{if } j_2 + r \geq j_3 + s \\ \begin{array}{c} .5\pi_1^2 e_{i_1} \Phi^{s-j_1} V_\Sigma (j_2 + r - s - 1) \Phi^{s-j_1\prime} e'_{i_1} + .5 \left(\pi_1 e_{i_1} \Phi^{j_2+r-j_1} + \pi_2 e_{i_2}\right) \\ \times V_\Sigma (j_3 + s - j_2 - r - 1) \left(\pi_1 e_{i_1} \Phi^{j_2+r-j_1} + \pi_2 e_{i_2}\right)' \end{array} & \text{if } j_3 + s \geq j_2 + r \end{Bmatrix}$$

In the case $j_3+s>s-1>j_1\geq j_2+r>r-1$, it is

$$\sigma_r^2\exp\begin{pmatrix} .5V_{\alpha\pi}\left(\rho_3,s-j_1-1,s-j_2-r-1,-\right)+.5V_{\alpha\pi}\left(\rho_4,s-j_1-2,s-j_2-r-2,-\right)\\ +.5\pi_2^2e_{i_2}V_\Sigma\left(j_1-j_2-r-1\right)e_{i_2}'\\ +.5\left(\pi_1e_{i_1}+\pi_2e_{i_2}\Phi^{j_1-j_2-r}\right)V_\Sigma\left(s-j_1-3\right)\left(\pi_1e_{i_1}+\pi_2e_{i_2}\Phi^{j_1-j_2-r}\right)'\\ +.5\left(\pi_1e_{i_1}\Phi^{s-j_1}+\pi_2e_{i_2}\Phi^{s-j_2-r}\right)V_\Sigma\left(j_3-1\right)\left(\pi_1e_{i_1}\Phi^{s-j_1}+\pi_2e_{i_2}\Phi^{s-j_2-r}\right)'\end{pmatrix}$$

In the case $j_3+s>j_2+r=s-1>j_1\geq r-1$, it is

$$\begin{Bmatrix} \sigma_r^2\exp\left(.5\pi_1^2e_{i_1}V_\Sigma\left(s-j_1-3\right)e_{i_1}'\right) & \text{if } j_1>r-1\\ \sigma_{r=1}^2\exp\left(.5V_{\alpha\pi}\left(\rho_1,0,-,-\right)+.5\pi_1^2e_{i_1}\Phi V_\Sigma\left(s-j_1-4\right)\Phi'e_{i_1}'\right) & \text{if } j_1=r-1\end{Bmatrix}$$
$$\times\exp\begin{pmatrix} .5V_{\alpha\pi}\left(\rho_4,s-j_1-2,-,-\right)'+.5\left(\rho_3\left(1,\alpha\right)+\pi_1e_{i_1}\Phi^{j_2+r-j_1}+\pi_2e_{i_2}\right)\\ \times V_\Sigma\left(j_3+s-j_2-r-1\right)\left(\rho_3\left(1,\alpha\right)+\pi_1e_{i_1}\Phi^{j_2+r-j_1}+\pi_2e_{i_2}\right)'\end{pmatrix}$$

In the case $j_3+s>s-1>j_2+r>j_1\geq r-1$, it is

$$\begin{Bmatrix} \sigma_r^2\exp\left(.5\pi_1^2e_{i_1}V_\Sigma\left(j_2+r-j_1-1\right)e_{i_1}'\right) & \text{if } j_1>r-1\\ \sigma_{r=1}^2\exp\left(.5\left(\rho_1\left(1,\alpha\right)+\pi_1e_{i_1}\right)V_\Sigma\left(j_2\right)\left(\rho_1\left(1,\alpha\right)+\pi_1e_{i_1}\right)'\right) & \text{if } j_1=r-1\end{Bmatrix}$$
$$\times\exp\begin{pmatrix} .5V_{\alpha\pi}\left(\rho_3,s-j_1-1,s-j_2-r-1,-\right)+.5V_{\alpha\pi}\left(\rho_4,s-j_1-2,s-j_2-r-2,-\right)\\ +.5\left(\pi_1e_{i_1}\Phi^{j_2+r-j_1}+\pi_2e_{i_2}\right)V_\Sigma\left(s-j_2-r-3\right)\left(\pi_1e_{i_1}\Phi^{j_2+r-j_1}+\pi_2e_{i_2}\right)'\\ +.5\left(\pi_1e_{i_1}\Phi^{s-j_1}+\pi_2e_{i_2}\Phi^{s-j_2-r}\right)V_\Sigma\left(j_3-1\right)\left(\pi_1e_{i_1}\Phi^{s-j_1}+\pi_2e_{i_2}\Phi^{s-j_2-r}\right)\end{pmatrix}$$

In the case $j_3+s>s-1>j_2+r>r-1>j_1$, it is

$$\sigma_-^2\exp\begin{pmatrix} .5V_{\alpha\pi}\left(\rho_1,r-j_1-1,-,-\right)+.5V_{\alpha\pi}\left(\rho_2,r-j_1-2,-,-\right)\\ +.5V_{\alpha\pi}\left(\rho_3,s-j_1-1,s-j_2-r-1,-\right)+.5V_{\alpha\pi}\left(\rho_4,s-j_1-2,s-j_2-r-2,-\right)\\ +.5\pi_1^2e_{i_1}V_\Sigma\left(r-j_1-3\right)e_{i_1}'+.5\pi_1^2e_{i_1}\Phi^{r-j_1}V_\Sigma\left(j_2-1\right)\Phi^{r-j_1\prime}e_{i_1}'\\ +.5\left(\pi_1e_{i_1}\Phi^{j_2+r-j_1}+\pi_2e_{i_2}\right)V_\Sigma\left(s-j_2-r-3\right)\left(\pi_1e_{i_1}\Phi^{j_2+r-j_1}+\pi_2e_{i_2}\right)'\\ +.5\left(\pi_1e_{i_1}\Phi^{s-j_1}+\pi_2e_{i_2}\Phi^{s-j_2-r}\right)V_\Sigma\left(j_3-1\right)\left(\pi_1e_{i_1}\Phi^{s-j_1}+\pi_2e_{i_2}\Phi^{s-j_2-r}\right)'\end{pmatrix}$$

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