‘Missing mean does no harm to volatility!’ by Stanislav Anatolyev and Irina Tarasyuk: analytical derivations

Moments of innovations

Denote $m_k = E\epsilon_t^k$ for integer k. Due to the distributional symmetry of ϵ_t, the odd moments of ϵ_t equal to zero, in particular $m_1 = m_3 = 0$ and $\mathbb{E}\epsilon_{t-1}\epsilon_t^2 = 0$. It is also easy to check that $\mathbb{E}\sigma_{t-1}^2\sigma_t^2\epsilon_{t-1} = \mathbb{E}\sigma_{t-1}^2\epsilon_{t-1}\sigma_t^4 = 0$ whenever these moments exist. Next note that

$$m_2 = \mathbb{E}\epsilon_t^2 = \sigma^2, \quad m_4 = \mathbb{E}\sigma_t^4 z_t^4 = 3\mathbb{E}\sigma_t^4, \quad m_6 = \mathbb{E}\sigma_t^6 z_t^6 = 15\mathbb{E}\sigma_t^6, \quad m_8 = \mathbb{E}\sigma_t^8 z_t^8 = 105\mathbb{E}\sigma_t^8.$$

For fourth moments to exist we need that $\alpha \in \left[0, 1/\sqrt{3}\right)$:

$$\begin{aligned} m_4 &= 3\mathbb{E}\left(\omega + \alpha\epsilon_{t-1}^2\right)^2 = 3\omega^2 + 6\omega\alpha m_2 + 3\alpha^2 m_4 \\ &= 3\sigma^4\frac{1-\alpha^2}{1-3\alpha^2}, \\ \mathbb{E}\epsilon_{t-1}^2\epsilon_t^2 &= \mathbb{E}\epsilon_{t-1}^2\sigma_t^2 = \mathbb{E}\epsilon_{t-1}^2\left(\omega + \alpha\epsilon_{t-1}^2\right) = \omega m_2 + \alpha m_4 \\ &= \sigma^4\frac{(1-\alpha)(1+3\alpha)}{1-3\alpha^2}. \end{aligned}$$

For sixth moments to exist we need that $\alpha \in \left[0, 1/\sqrt[3]{15}\right)$:

$$\begin{aligned} \mathbb{E}\sigma_t^6 &= \mathbb{E}\left(\omega + \alpha\epsilon_{t-1}^2\right)^3 = \omega^3 + 3\omega^2\alpha m_2 + 3\omega\alpha^2 m_4 + \alpha^3 m_6 \\ &= \sigma^6\frac{(1-\alpha)^2\left(1+2\alpha+6\alpha^2+3\alpha^3\right)}{(1-3\alpha^2)(1-15\alpha^3)}, \\ \mathbb{E}\epsilon_{t-1}^2\sigma_t^4 &= \mathbb{E}\epsilon_{t-1}^2\left(\omega + \alpha\epsilon_{t-1}^2\right)^2 = \omega^2 m_2 + 6\alpha\omega\mathbb{E}\sigma_{t-1}^4 + 15\alpha^2\mathbb{E}\sigma_{t-1}^6 \\ &= \sigma^6\frac{(1-\alpha)^2\left(1+6\alpha+18\alpha^2+15\alpha^3\right)}{(1-3\alpha^2)(1-15\alpha^3)}, \\ \mathbb{E}\sigma_{t-1}^4\sigma_t^2 &= \mathbb{E}\sigma_{t-1}^4\left(\omega + \alpha\epsilon_{t-1}^2\right) = \omega\mathbb{E}\sigma_{t-1}^4 + \alpha\mathbb{E}\sigma_{t-1}^6 \\ &= \sigma^6\frac{(1-\alpha)^2\left(1+2\alpha+2\alpha^2-9\alpha^3-12\alpha^4\right)}{(1-3\alpha^2)(1-15\alpha^3)}. \end{aligned}$$

For eighth moments to exist we need that $\alpha \in \left[0, 1/\sqrt[4]{105}\right)$:

$$\begin{aligned} \mathbb{E}\sigma_t^8 &= \mathbb{E}\left(\omega + \alpha\epsilon_{t-1}^2\right)^4 = \omega^4 + 4\alpha\omega^3 m_2 + 6\alpha^2\omega^2 m_4 + 60\alpha^3\omega\mathbb{E}\sigma_{t-1}^6 + 105\alpha^4\mathbb{E}\sigma_{t-1}^8 \\ &= \sigma^8\frac{(1-\alpha)^3\left(1+3\alpha+15\alpha^2+54\alpha^3+75\alpha^4+135\alpha^5+45\alpha^6\right)}{(1-3\alpha^2)(1-15\alpha^3)(1-105\alpha^4)}, \\ \mathbb{E}\sigma_{t-1}^4\sigma_t^4 &= \mathbb{E}\sigma_{t-1}^4\left(\omega + \alpha\epsilon_{t-1}^2\right)^2 = \omega^2\mathbb{E}\sigma_{t-1}^4 + 2\alpha\omega\mathbb{E}\sigma_{t-1}^6 + 3\alpha^2\mathbb{E}\sigma_{t-1}^8 \\ &= \sigma^8\frac{(1-\alpha)^3\left(1+3\alpha+7\alpha^2+6\alpha^3-69\alpha^4-153\alpha^5-195\alpha^6+720\alpha^7+1080\alpha^8\right)}{(1-3\alpha^2)(1-15\alpha^3)(1-105a^4)}. \end{aligned}$$

'Lost constant' scenario

As $r_t|F_{t-1} \sim N(\mu, \sigma_t^2)$, the ingredients of the formula for α' are:

$$\begin{aligned}
\mathbb{E}r_t^2 &= \mu^2 + \sigma^2, \\
\mathbb{E}r_t^3 &= \mathbb{E}(\mu + \epsilon_t)^3 = \mu^3 + 3\mu\sigma^2, \\
\mathbb{E}r_t^4 &= \mathbb{E}(\mu + \epsilon_t)^4 = \mu^4 + 6\mu^2 m_2 + m_4 = \mu^4 + 6\mu^2\sigma^2 + \frac{3\sigma^4(1-\alpha^2)}{1-3\alpha^2}, \\
\mathbb{E}r_t^2 r_{t-1}^2 &= \mu^4 + \mu^2\mathbb{E}\epsilon_{t-1}^2 + \mu^2\mathbb{E}\epsilon_t^2 + \mathbb{E}\epsilon_t^2\epsilon_{t-1}^2 = \mu^4 + 2\mu^2\sigma^2 + \sigma^4\frac{(1-\alpha)(1+3\alpha)}{1-3\alpha^2}.
\end{aligned}$$

Hence, a closed-form expression for α' is

$$\alpha' = \frac{\alpha}{1 + 2s(1-3\alpha^2)},$$

where $s = (\mu/\sigma)^2$ is a scale parameter.

'Lost seasonality' scenario

Note the α' is defined in terms of even moments of r_t only. Because ϵ_t has a symmetric distribution, the expression for α' is the same as under the 'lost constant' scenario.

'Lost autoregression' scenario

When $r_t|F_{t-1} \sim N(\rho r_{t-1}, \sigma_t^2)$, the odd moments are zero, and the ingredients of the formula for α' are:

$$\begin{aligned}
\mathbb{E}r_t^2 &= \rho^2\mathbb{E}r_{t-1}^2 + \sigma^2 = \frac{\sigma^2}{1-\rho^2}, \\
\mathbb{E}r_{t-1}^2\epsilon_t^2 &= \omega\mathbb{E}r_{t-1}^2 + \alpha\mathbb{E}r_{t-1}^2\epsilon_{t-1}^2 = \omega\mathbb{E}r_{t-1}^2 + \alpha\mathbb{E}(\rho r_{t-2} + \epsilon_{t-1})^2\epsilon_{t-1}^2 \\
&= \omega\mathbb{E}r_{t-1}^2 + \alpha\rho^2\mathbb{E}r_{t-2}^2\epsilon_{t-1}^2 + \alpha m_4 \\
&= \frac{\omega\mathbb{E}r_t^2 + \alpha m_4}{1-\alpha\rho^2}, \\
\mathbb{E}r_t^2 r_{t-1}^2 &= \mathbb{E}\left(\rho^2 r_{t-1}^2 + 2\rho r_{t-1}\epsilon_t + \epsilon_t^2\right) r_{t-1}^2 = \rho^2\mathbb{E}r_{t-1}^4 + \mathbb{E}r_{t-1}^2\epsilon_t^2, \\
\mathbb{E}r_t^4 &= \mathbb{E}(\rho r_{t-1} + \epsilon_t)^4 = \rho^4\mathbb{E}r_{t-1}^4 + 6\rho^2\mathbb{E}r_{t-1}^2\epsilon_t^2 + m_4.
\end{aligned}$$

Collecting the pieces together, a closed-form expression for α' is

$$\alpha' = \frac{\alpha + (1-\alpha)s + (1+4\alpha-6\alpha^2)s^2 - 6\alpha(1-\alpha^2)s^3}{1 + (1+5\alpha-6\alpha^2)s - \alpha(7-6\alpha^2)s^2},$$

where $s = \rho^2$ is a scale parameter.

'Lost risk premium' scenario

When $r_t|F_{t-1} \sim N\left(\delta\sigma_t^2, \sigma_t^2\right)$, the mean equals $\mathbb{E}r_t = \delta\sigma^2$. Then, the ingredients of the formula for α' are:

$$\begin{aligned}
\mathbb{E}r_t^2 &= \mathbb{E}\left((\delta\sigma_t^2)^2 + \sigma_t^2\right) = \sigma^2 + \delta^2\mathbb{E}\sigma_t^4, \\
\mathbb{E}r_t^3 &= \mathbb{E}\left(\delta\sigma_t^2 + \epsilon_t\right)^3 = \delta^3\mathbb{E}\sigma_t^6 + 3\delta\mathbb{E}\sigma_t^4, \\
\mathbb{E}r_t^4 &= \mathbb{E}\left(\delta\sigma_t^2 + \epsilon_t\right)^4 = \delta^4\mathbb{E}\sigma_t^8 + 6\delta^2\mathbb{E}\sigma_t^6 + m_4, \\
\mathbb{E}r_{t-1}^2 r_t^2 &= \mathbb{E}\left(\delta\sigma_{t-1}^2 + \epsilon_{t-1}\right)^2\left(\sigma_t^2 + \delta^2\sigma_t^4\right) \\
&= \delta^2\mathbb{E}\sigma_{t-1}^4\sigma_t^2 + \mathbb{E}\epsilon_{t-1}^2\sigma_t^2 + \delta^4\mathbb{E}\sigma_{t-1}^4\sigma_t^4 + \delta^2\mathbb{E}\epsilon_{t-1}^2\sigma_t^4.
\end{aligned}$$

Collecting the pieces together, a closed-form expression for α' is $\alpha' = N_\alpha/D_\alpha$, where

$$\begin{aligned}
N_\alpha &= \alpha\left(-1 + 3\alpha^2 + 15\alpha^3 + 105\alpha^4 - 45\alpha^5 - 315\alpha^6 - 1575\alpha^7 + 4725\alpha^9\right) \\
&- 2\alpha s\left(1-\alpha\right)\left(1+\alpha^2\right)\left(1-3\alpha^2\right)\left(1-105\alpha^4\right)\left(1+3\alpha+3\alpha^2\right) \\
&- 4\alpha^3 s^2\left(1-\alpha\right)^2\left(1 + 6\alpha + 18\alpha^2 + 36\alpha^3 - 51\alpha^4 - 333\alpha^5 - 675\alpha^6 - 135\alpha^7 + 405\alpha^8\right), \\
D_\alpha &= -1 - 2s + 3\alpha^2\left(1 + 5\alpha\left(1-3\alpha^2\right)\left(1 + 7\alpha - 105\alpha^4\right)\right) \\
&+ 2\alpha^2 s\left(-2 + 3\alpha\left(1 - \left(1-\alpha\right)\alpha\left(-40 + \alpha\left(-38 + 3\alpha\left(-36 + 35\alpha^2\left(5 + 3\alpha\left(1+\alpha\right)\right)\right)\right)\right)\right)\right) \\
&+ 4\alpha^2 s^2\left(1-\alpha\right)^2\left(-1 - 6\alpha - 15\alpha^2 - 21\alpha^3 + 6\alpha^4 + 225\alpha^5 + 360\alpha^6 + 180\alpha^7\right).
\end{aligned}$$

where $s = (\delta\sigma)^2$ is a scale parameter.