Autoregression and redundant instruments

Stanislav Anatolyev*

April 1, 2002

Problem

Consider a zero mean stationary autoregressive model of order k with IID innovations having variance σ^2:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \cdots + \rho_k y_{t-k} + \varepsilon_t.$$

It is well known that the efficient GMM estimator of $\rho = (\rho_1\ \rho_2\ \cdots\ \rho_k)'$ based on the instrumental vector $z_t = (y_{t-1}\ y_{t-2}\ \cdots\ y_{t-k}\ y_{t-k-1}\ \cdots\ y_{t-\ell})'$ consisting of the last $\ell > k$ lags of y_t, effectively exploits information in the most recent k lags of y_t (see, for example, Kim, Qian and Schmidt, 1999). In other words, the instruments $y_{t-k-1}, \cdots, y_{t-\ell}$ are redundant (see Breusch, Qian, Schmidt and Wyhowski, 1999) given $y_{t-1}, \cdots, y_{t-k}$.

Prove the following more general proposition: when one uses the instrumental vector $z_t = (y_{t-p}\ y_{t-p-1}\ \cdots\ y_{t-p-k+1}\ y_{t-p-k}\ \cdots\ y_{t-p-\ell+1})'$ for $p \geq 1$, the instruments $y_{t-p-k}, \cdots, y_{t-p-\ell+1}$ are redundant given $y_{t-p}, \cdots, y_{t-p-k+1}$.

References

Breusch, T., H. Qian, P. Schmidt & D. Wyhowski (1999) Redundancy of moment conditions. *Journal of Econometrics* 91, 89–111.

Kim, Y., H. Qian & P. Schmidt (1999) Efficient GMM and MD estimation of autoregressive models. *Economics Letters* 62, 265–270.

*New Economic School, Nakhimovsky prospect, 47, room 1721, Moscow, 117418, Russia. E-mail: sanatoly@nes.ru

Suggested Solution

The vector of regressors is $x_t = (y_{t-1}\, y_{t-2}\, \cdots\, y_{t-k})'$. Let $\gamma_j = \gamma_{-j} = \mathrm{E}[y_t y_{t-j}]$ and $\Gamma_j = \left(\gamma_j\, \gamma_{j-1}\, \cdots\, \gamma_{j-\ell+1}\right)'$. The matrix of cross-covariances of x_t and z_t is

$$\begin{aligned}
\mathrm{Q}_{xz} &= \begin{pmatrix} \gamma_{p-1} & \gamma_p & \cdots & \gamma_{p+k-2} & \cdots & \gamma_{p+\ell-2} \\ \gamma_{p-2} & \gamma_{p-1} & \cdots & \gamma_{p+k-3} & \cdots & \gamma_{p+\ell-3} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_{p-k} & \gamma_{p-k+1} & \cdots & \gamma_{p-1} & \cdots & \gamma_{p+\ell-k-1} \end{pmatrix} \\
&= \left(\mathrm{I}_k\; \mathrm{O}_{k\times(\ell-k)}\right)\left(\Gamma_{p-1}\, \Gamma_p\, \cdots\, \Gamma_{p+\ell-2}\right),
\end{aligned}$$

where I_m denotes $m\times m$ identity matrix, and $\mathrm{O}_{m_1\times m_2}$ – zero $m_1\times m_2$ matrix. The covariance matrix of $z_t\varepsilon_t$ is

$$\mathrm{V}_{z\varepsilon} = \sigma^2 \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{\ell-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{\ell-1} & \gamma_{\ell-2} & \cdots & \gamma_0 \end{pmatrix} = \sigma^2\left(\Gamma_0\, \Gamma_1\, \cdots\, \Gamma_{\ell-1}\right).$$

The efficient GMM estimator based on the instrumental vector z_t effectively uses the instrument optimal in the class of linear transformations of z_t. This optimal instrument is $\mathrm{Q}_{xz}\mathrm{V}_{z\varepsilon}^{-1}z_t$ (see, for example, West, 2001). We will show that in the matrix $\mathrm{Q}_{xz}\mathrm{V}_{z\varepsilon}^{-1}$ the right $k\times(\ell-k)$ submatrix is zero, so that the optimal combination of elements of z_t involves only first k entries.

Recall that the Yule–Walker equations for an AR(k) model contain the following recursion:

$$\gamma_j = \rho_1\gamma_{j-1} + \rho_2\gamma_{j-2} + \cdots + \rho_k\gamma_{j-k}, \quad j \geq 1.$$

This implies that

$$\begin{aligned}
\left(\Gamma_{p-1}\, \Gamma_p\, \cdots\, \Gamma_{p+\ell-2}\right) &= \mathrm{P}\left(\Gamma_{p-2}\, \Gamma_{p-1}\, \cdots\, \Gamma_{p+\ell-3}\right) \\
&= \cdots \\
&= \mathrm{P}^{p-1}\left(\Gamma_0\, \Gamma_1\, \cdots\, \Gamma_{\ell-1}\right),
\end{aligned}$$

where

$$\mathrm{P} = \begin{pmatrix} \rho' & 0'_{\ell-1-k} & \\ & \mathrm{I}_{\ell-1} & 0_\ell \end{pmatrix},$$

and 0_m denotes zero $m\times 1$ vector. Thus

$$\begin{aligned}
\mathrm{Q}_{xz} &= \left(\mathrm{I}_k\; \mathrm{O}_{k\times(\ell-k)}\right)\left(\Gamma_{p-1}\, \Gamma_p\, \cdots\, \Gamma_{p+\ell-2}\right) \\
&= \sigma^{-2}\left(\mathrm{I}_k\; \mathrm{O}_{k\times(\ell-k)}\right)\mathrm{P}^{p-1}\mathrm{V}_{z\varepsilon}.
\end{aligned}$$

It follows that $\mathrm{Q}_{xz}\mathrm{V}_{z\varepsilon}^{-1}$ equals $\sigma^{-2}\left(\mathrm{I}_k\; \mathrm{O}_{k\times(\ell-k)}\right)\mathrm{P}^{p-1}$, a matrix whose right $k\times(\ell-k)$ submatrix is indeed zero.

Reference

West, K.D. (2001) On Optimal Instrumental Variables Estimation of Stationary Time Series Models. *International Economic Review* 42, 1043–1050.