Durbin–Watson Statistic and Random Individual Effects

Stanislav Anatolyev*

March 11, 2002

Problem

Consider the standard one-way error component model with random effects (Baltagi, 2001):

$$y_{it} = x_{it}'\beta + \mu_i + v_{it}, \quad i = 1, \cdots, n, \quad t = 1, \cdots, T, \tag{1}$$

where β is $k \times 1$, μ_i are random individual effects, $\mu_i \sim IID(0, \sigma_\mu^2)$, v_{it} are idiosyncratic shocks, $v_{it} \sim IID(0, \sigma_v^2)$, and μ_i and v_{it} are independent of x_{it} for all i and t and mutually. The equations are arranged so that the index t is faster than the index i. Consider running OLS on the original regression (1); running OLS on the Within regression

$$y_{it} - \bar{y}_{i\cdot} = (x_{it} - \bar{x}_{i\cdot})'\beta + v_{it} - \bar{v}_{i\cdot}, \quad i = 1, \cdots, n, \quad t = 1, \cdots, T, \tag{2}$$

where $\bar{z}_{i\cdot} = T^{-1} \sum_{t=1}^{T} z_{it}$ for $z = y, x, v$; running OLS on the Between regression

$$\bar{y}_{i\cdot} = \bar{x}_{i\cdot}'\beta + \mu_i + \bar{v}_{i\cdot}, \quad i = 1, \cdots, n, \quad t = 1, \cdots, T, \tag{3}$$

with T replications of the equation for each individual i; and running OLS on the GLS-transformed regression

$$y_{it} - \hat{\theta}\bar{y}_{i\cdot} = (x_{it}' - \hat{\theta}\bar{x}_{i\cdot})'\beta + (1 - \hat{\theta})\mu_i + v_{it} - \hat{\theta}\bar{v}_{i\cdot}, \quad i = 1, \cdots, n, \quad t = 1, \cdots, T, \tag{4}$$

where $\hat{\theta}$ is a consistent (as $n \to \infty$ and T stays fixed) estimate of $\theta = 1 - \sigma_v / \sqrt{\sigma_v^2 + T\sigma_\mu^2}$. When each OLS estimate is obtained using a typical regression package, the Durbin–Watson statistic is provided among the regression output. Derive the probability limits of the four Durbin–Watson statistics, as $n \to \infty$ and T stays fixed. Using the obtained result, propose an asymptotic test for individual effects based on the Durbin–Watson statistic.

Reference

Baltagi, B.H. (2001) Econometric Analysis of Panel Data. New York: John Wiley & Sons.

*New Economic School, Nakhimovsky prospect, 47, room 1721, Moscow, 117418, Russia. E-mail: sanatoly@nes.ru

Suggested Solution

In all regressions, the residuals consistently estimate corresponding regression errors. Therefore, to find a probability limit of the Durbin–Watson statistic, it suffices to compute the variance and first-order autocovariance of the errors across the stacked equations:

$$\operatorname*{p\,lim}_{n\to\infty} DW = 2\left(1-\frac{\varrho_1}{\varrho_0}\right),$$

where

$$\varrho_0 = \operatorname*{p\,lim}_{n\to\infty}\frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}u_{it}^2, \quad \varrho_1 = \operatorname*{p\,lim}_{n\to\infty}\frac{1}{nT}\sum_{t=2}^{T}\sum_{i=1}^{n}u_{it}u_{i,t-1},$$

and u_{it}'s denote regression errors. Note that the errors are uncorrelated where the index i switches between individuals, hence summation from $t=2$ in ϱ_1.

Consider the original regression (1) where $u_{it}=\mu_i+v_{it}$. Then $\varrho_0=\sigma_v^2+\sigma_\mu^2$ and

$$\varrho_1 = \frac{1}{T}\sum_{t=2}^{T}\operatorname*{p\,lim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i+v_{it}\right)\left(\mu_i+v_{i,t-1}\right) = \frac{T-1}{T}\sigma_\mu^2.$$

Thus

$$\operatorname*{p\,lim}_{n\to\infty} DW_{OLS} = 2\left(1-\frac{T-1}{T}\frac{\sigma_\mu^2}{\sigma_v^2+\sigma_\mu^2}\right) = 2\frac{T\sigma_v^2+\sigma_\mu^2}{T\left(\sigma_v^2+\sigma_\mu^2\right)}.$$

Consider the Within regression (2) where $u_{it}=v_{it}-\bar{v}_{i\cdot}$. Then

$$\varrho_0 = \frac{1}{T}\sum_{t=1}^{T}\operatorname*{p\,lim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{T-1}{T}v_{it}-\frac{1}{T}\sum_{\tau\neq t}v_{i\tau}\right)^2 = \frac{T-1}{T}\sigma_v^2$$

and

$$\begin{aligned}\varrho_1 &= \frac{1}{T}\sum_{t=2}^{T}\operatorname*{p\,lim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{T-1}{T}v_{it}-\frac{1}{T}v_{i,t-1}-\frac{1}{T}\sum_{\substack{\tau\neq t\\ \tau\neq t-1}}v_{i\tau}\right)\left(\frac{T-1}{T}v_{i,t-1}-\frac{1}{T}v_{it}-\frac{1}{T}\sum_{\substack{\tau\neq t\\ \tau\neq t-1}}v_{i\tau}\right)\\ &= -\frac{T-1}{T^2}\sigma_v^2.\end{aligned}$$

Thus

$$\operatorname*{p\,lim}_{n\to\infty} DW_{Within} = 2\frac{T+1}{T}.$$

Consider the Between regression (3) where $u_{it} = \mu_i + \bar{v}_{i\cdot}$. Then

$$\varrho_0 = \frac{1}{T}\sum_{t=1}^{T} \operatorname*{p\,lim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (\mu_i + \bar{v}_{i\cdot})^2 = \sigma_\mu^2 + \frac{1}{T}\sigma_v^2$$

and

$$\varrho_1 = \frac{1}{T}\sum_{t=2}^{T} \operatorname*{p\,lim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (\mu_i + \bar{v}_{i\cdot})^2 = \frac{T-1}{T}\left(\sigma_\mu^2 + \frac{1}{T}\sigma_v^2\right).$$

Thus

$$\operatorname*{p\,lim}_{n\to\infty} DW_{Between} = \frac{2}{T}.$$

The GLS-transformation orthogonalizes the errors, therefore

$$\operatorname*{p\,lim}_{n\to\infty} DW_{GLS} = 2.$$

Since all computed probability limits except that for DW_{OLS} do not depend on the variance components, the only way to construct an asymptotic test of $H_0 : \sigma_\mu^2 = 0$ vs. $H_A : \sigma_\mu^2 > 0$ is by using DW_{OLS}. Under H_0, $\sqrt{nT}\,(DW_{OLS} - 2) \xrightarrow{d} N(0, 4)$ as $n \to \infty$ (estimation of β does not affect the limiting distribution). Under H_A, $\operatorname*{p\,lim}_{n\to\infty} DW_{OLS} < 2$. Hence a one-sided asymptotic test for $\sigma_\mu^2 = 0$ for a given level α is:

$$\text{Reject if } DW_{OLS} < 2\left(1 + \frac{z_\alpha}{\sqrt{nT}}\right),$$

where z_α is the α-quantile of the standard normal distribution.