Redundancy of lagged regressors in a conditionally heteroskedastic time series regression

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Problem

Consider the following stationary time series regression:

$$y_t = \beta x_t + e_t, \quad E\left[e_t | x_t, x_{t-1}, \cdots\right] = 0,$$

where all variables are scalars, and the error e_t is conditionally heteroskedastic with the following form of heteroskedasticity:

$$E\left[e_t^2 | x_t, x_{t-1}, \cdots\right] = \omega + \lambda \left(x_t - \mu\right)^2, \quad \omega > 0, \quad \lambda \geq 0.$$

The object of estimation is β. Usually, under conditional heteroskedasticity the use of lagged values of regressors as instruments increases the efficiency of GMM estimation in comparison with OLS estimation (see, for example, Broze, Francq and Zakoïan, 2001, West 2001). This problem shows that it may not be necessarily so.

Assume that the regressor x_t can be represented as $x_t = \sum_{i=0}^{\infty} \varphi_i \eta_{t-i}$, where η_t's are IID standard normal. Also assume that the parameters are constrained so that all variables have finite fourth moments. Show that the OLS estimator is at least as efficient as any GMM estimator that uses an arbitrary fixed number of instruments from the list $\{x_t, x_{t-1}, x_{t-2}, \cdots\}$.

References

Broze, L., C. Francq & J.-M. Zakoïan (2001) Non-redundancy of High Order Moment Conditions for Efficient GMM Estimation of Weak AR Processes. *Economics Letters* 71, 317–322.

West, K.D. (2001) On Optimal Instrumental Variables Estimation of Stationary Time Series Models. *International Economic Review* 42, 1043–1050.

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Suggested Solution

Let us consider the linear space of instruments $\mathcal{Z}_t$ spanned by the present and lagged x_t. Any GMM estimator described in the problem is asymptotically at most as efficient as the GMM estimator based on the instrument optimal relative to $\mathcal{Z}_t$. Let the representation of the optimal instrument be $x_t^* = \sum_{i=0}^{\infty} g_i \eta_{t-i}$, and denote $\sigma_x^2 \equiv \sum_{j=0}^{\infty} \varphi_j^2$ and $\tau \equiv E[\eta_t^4] - 1$ (in our case, $\tau = 2$). The optimality condition (Hansen, 1985, West, 2001) is

$$\forall k \geq 0 \qquad E\left[\eta_{t-k} x_t\right] = E\left[\eta_{t-k} x_t^* e_t^2\right]. \tag{1}$$

The left hand side in (1) is φ_k. The right hand side in (1) is:

$$\begin{aligned} E\left[\eta_{t-k} x_t^* e_t^2\right] &= E\left[\eta_{t-k}\left(\sum_{i=0}^{\infty} g_i \eta_{t-i}\right)\left(\omega + \lambda\left(\sum_{j=0}^{\infty} \varphi_j \eta_{t-j} - \mu\right)^2\right)\right] \\ &= \left(\omega + \lambda\left(\mu^2 + \sigma_x^2 + \tau\varphi_k^2\right)\right) g_k + 2\lambda\varphi_k \sum_{i=0,\, i\neq k}^{\infty} \varphi_i g_i. \end{aligned}$$

Therefore the system (1) can be written in matrix form as follows:

$$\Phi = \mathrm{SG}, \tag{2}$$

where $\Phi \equiv \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_k \\ \vdots \end{bmatrix}$, $\mathrm{G} \equiv \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_k \\ \vdots \end{bmatrix}$, $\mathrm{S} \equiv \begin{bmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,k} & \cdots \\ S_{1,0} & S_{1,1} & \cdots & S_{1,k} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ S_{k,0} & S_{k,1} & \cdots & S_{k,k} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}$,

and $S_{k,k} = \omega + \lambda\left(\mu^2 + \sigma_x^2 + \tau\varphi_k^2\right)$, $S_{k,m} = 2\lambda\varphi_k\varphi_m$, $k \geq 0$, $m \geq 0$, $k \neq m$. Note that

$$\mathrm{S} = \mathrm{diag}\left(\omega + \lambda\left(\mu^2 + \sigma_x^2 + (\tau - 2)\,\varphi_k^2\right)\right) + 2\lambda\Phi\Phi'.$$

When $\tau = 2$, the solution to (2) is proportional to Φ. Indeed, for $\tau = 2$ and $\mathrm{G} = \delta\Phi$ for some scalar δ we have

$$\begin{aligned} \mathrm{SG} &= \left(\mathrm{diag}\left(\omega + \lambda\mu^2 + \lambda\sigma_x^2\right) + 2\lambda\Phi\Phi'\right)\delta\Phi \\ &= \delta\left(\omega + \lambda\mu^2 + \lambda\sigma_x^2\right)\Phi + 2\lambda\delta\Phi\left(\Phi'\Phi\right) \\ &= \delta\left(\omega + \lambda\mu^2 + 3\lambda\sigma_x^2\right)\Phi, \end{aligned}$$

so (2) is satisfied with $\delta = \left(\omega + \lambda\mu^2 + 3\lambda\sigma_x^2\right)^{-1}$.

The fact that G proportional to Φ implies that the optimal instrument is a multiple of the regressor. Hence the OLS estimator is an optimal instrumental variables estimator and is efficient in the class considered. Note that our derivation takes advantage of the fact that the distribution of η_t is symmetric with $\tau = 2$. Also, the conclusion does not necessarily hold if the skedastic function is not quadratic in x_t.

References

Hansen, L.P. (1985) A Method for Calculating Bounds on the Asymptotic Variance-Covariance Matrices of Generalized Method of Moments Estimators. *Journal of Econometrics* 30, 203–228.

West, K.D. (2001) On Optimal Instrumental Variables Estimation of Stationary Time Series Models. *International Economic Review* 42, 1043–1050.