An algorithm for constructing high dimensional distributions from distributions of lower dimension*

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Abstract

We propose a new sequential procedure for estimating multivariate distributions in cases when conventional maximum likelihood has too many parameters and is therefore inaccurate or non-operational. The procedure constructs a multivariate distribution and its pseudo-likelihood sequentially, in each step using lower-dimensional distributions with a small number of parameters. In an application, the procedure provides excellent fit when the dimension is moderate, and remains operational when the conventional method fails.

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1 Introduction

Consider the problem of constructing a high dimensional distribution. As an example, suppose we wish to estimate a d-dimensional Student-t distribution. The problem has at least $d(d-1)/2$ parameters. The conventional approach is to construct a joint log-density from this d-dimensional distribution and use it in a maximum likelihood (ML) routine. However, for large d and moderate sample sizes, the likelihood is highly unstable, Hessians are near singular, estimates are inaccurate, and global convergence is hard to achieve.

One solution is to use copulas which have tighter parameterizations. However, the functional form of such copulas limits the nature of dependence they can accommodate (Nelsen, 2006, Section 4.6). Another solution is to use 'vine copulas' (Aas, Czado, Frigessi, and Bakken, 2009) when the d-variate density is decomposed into a product of up to $d(d-1)/2$ bivariate densities. However, there are still $O(d^2)$ parameters in the joint likelihood; in addition, the required ordering of components is rarely available, especially in the time series context. Yet another alternative is to use the factor copula approach (Oh and Patton, 2013). However, the joint density obtained lacks a close form; in addition, it is unclear whether the convolution of distributions imposed by the factor copula covers all classes of joint distributions one may wish to model.

The proposed method replaces the initial estimation problem with a sequence of bivariate problems. The procedure can be thought of as recovering the joint distribution from the distributions of all lower-dimensional sub-vectors comprising the original random vector. This provides sufficient flexibility as there are more degrees of freedom in choosing a parameterization in each step. The proposed estimator can be viewed as a traditional pseudo maximum likelihood estimator, but it is more flexible and works reasonably well in situations when the traditional ML fails.

2 The algorithm

In this section we describe the proposed algorithm, while in the next section we discuss its asymptotic properties.

Step 1. Estimate the marginals by fitting a suitable parametric distribution $\widehat{F}_j = F(\widehat{\theta}_j)$ for each $j = 1, ..., d$. This step involves d estimation problems.

Step 2. Using the $\widehat{F}_j$'s, estimate a bivariate distribution $\widehat{F}_{ij} = C^{(2)}(\widehat{F}_i, \widehat{F}_j; \widehat{\theta}_{ij})$ for each pair (i, j), where $C^{(2)}$ denotes a bivariate copula. There are $d(d-1)$ estimation problems in this step.

Step 3. Using the $\widehat{F}_j$'s and $\widehat{F}_{ij}$'s, estimate a trivariate distribution $C^{(3)}(\widehat{F}_i, \widehat{F}_{jk}; \widehat{\theta}_{ijk})$, for each combination of i and (j, k), where $C^{(3)}$ is a suitable *compounding function* capturing dependence between each element i and each disjoint pair (j,k). There are $d(d-1)(d-2)/2$ such combinations. Now, average $(\widehat{F}_i, \widehat{F}_{jk})$ over permutations of (i,j,k):

$$\widehat{F}_{ijk} = \frac{C^{(3)}\left(\widehat{F}_i, \widehat{F}_{jk}; \widehat{\theta}_{ijk}\right) + C^{(3)}\left(\widehat{F}_j, \widehat{F}_{ik}; \widehat{\theta}_{jik}\right) + C^{(3)}\left(\widehat{F}_k, \widehat{F}_{ij}; \widehat{\theta}_{kij}\right)}{3}.$$

Step m. Using the $\widehat{F}_j$'s and $\widehat{F}_{i_1,\ldots,j-1,j+1,\ldots,i_m}$, estimate an m-dimensional distribution of each m-tuple. There are $d!/(d-m)!(m-1)!$ possible combinations of $\widehat{F}_i$'s with disjoint $(m-1)$-variate marginals. Let $i_1 < i_2 < \ldots < i_m$, then obtain a model average estimate of the distribution for the $(i_1, i_2, \ldots, i_m)$-th m-tuple:

$$\widehat{F}_{i_1 i_2 \ldots i_m} = \frac{1}{m}\sum_{l=1}^{m} C^{(m)}(\widehat{F}_l, \widehat{F}_{i_1,\ldots,l-1,l+1,\ldots,i_m}; \widehat{\theta}_{l,i_1,\ldots,l-1,l+1,\ldots,i_m}),$$

where $C^{(m)}$ is an m-th order compounding function which is set to be a suitable asymmetric bivariate copula.

Step d. Estimate the d-variate distribution:

$$\widehat{F}_{12\ldots d} = \frac{1}{d}\sum_{l=1}^{d} C^{(d)}(\widehat{F}_l, \widehat{F}_{1,\ldots,l-1,l+1,\ldots,d}; \widehat{\theta}_{l,1,\ldots,l-1,l+1,\ldots,d}),$$

where $C^{(d)}$ is a d-th order compounding function. There are d such functions to be estimated.

3 Asymptotic properties

Let $\widehat{\boldsymbol{\theta}}$ contain all $\hat{\theta}$'s from Steps 1 to d. Then, by the Sklar (1959) theorem, the distribution $\widehat{F}_{12\ldots d}(x_1,\ldots,x_d)$ implies a d-copula $K(u_1,\ldots,u_d;\widehat{\boldsymbol{\theta}})$ and the corresponding estimator of density $\widehat{f}_{12\ldots d}(x_1,\ldots,x_d)$ implies a d-copula density $k(u_1,\ldots,u_d;\widehat{\boldsymbol{\theta}})$.[1] There is no guarantee that the m-th order compounding functions are also m-copulas, $m = 3,\ldots,d$, unless we use a compatible copula family.[2] However, the resulting estimator $\widehat{F}_{12\ldots d}$ is a continuous, non-decreasing, bounded d-variate function with range $[0,1]$, which *is* a distribution and thus implies a d-copula. The following result gives explicit formulas for the copula (density) implied by our estimator.

Proposition 1 *Let $\widehat{F}_m^{-1}(u_m), m = 1,\ldots,d$, denote the inverse of the marginal cdf $\widehat{F}_m$ from Step 1 and let $\widehat{f}_m$ denote the pdf corresponding to $\widehat{F}_m$. Then, the copula implied by $\widehat{F}_{12\ldots d}$ can be written*

[1] We denote the implied copula distribution and density functions by K and k, respectively, to distinguish them from the true copula distribution $C(u_1,\ldots,u_d)$ and true copula density $c(u_1,\ldots,u_d)$.

[2] There are several impossibility results concerning construction of high dimensional copulas by using lower dimensional copulas as argument of bivariate copulas (Quesada-Molina and Rodriguez-Lallena, 1994).

as follows:

$$
\begin{aligned}
K(u_1,\ldots,u_d;\widehat{\boldsymbol{\theta}}) &= \widehat{F}_{12\ldots d}(\widehat{F}_1^{-1}(u_1),\ldots,\widehat{F}_d^{-1}(u_d)),\\
k(u_1,\ldots,u_d;\widehat{\boldsymbol{\theta}}) &= \frac{\widehat{f}_{12\ldots d}(\widehat{F}_1^{-1}(u_1),\ldots,\widehat{F}_d^{-1}(u_d))}{\prod_{m=1}^{d}\widehat{f}_m(\widehat{F}_m^{-1}(u_m))}.
\end{aligned}
$$

It is clear from Proposition 1 that our algorithm provides an estimate of a flexible parametric d-variate *pseudo*-copula[3]. So the asymptotic properties of our estimator are basically the well-studied properties of copula-based *pseudo*- or *quasi*-ML estimator (Joe, 2005; Prokhorov and Schmidt, 2009). The following proposition summarizes these results, without proof.

Proposition 2 *The estimator $\widehat{\boldsymbol{\theta}}$ minimizes the Kullback-Leibler divergence criterion,*

$$
\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \mathbb{E} \ln \frac{c(u_1,\ldots,u_d)}{k(u_1,\ldots,u_d;\boldsymbol{\theta})},
$$

where c is the true copula density and expectation is with respect to the true distribution. Furthermore, under standard regularity conditions, $\widehat{\boldsymbol{\theta}}$ is consistent and asymptotically normal. If the true copula belongs to the family $k(u_1,\ldots,u_d;\boldsymbol{\theta})$, it is consistent for the true value of $\boldsymbol{\theta}$. If the copula family is misspecified, the convergence is to a pseudo-true value of $\boldsymbol{\theta}$, which minimizes the Kullback-Leibler distance.

Fundamentally, our algorithm uses the following form of the joint distribution:

$$
H(x_1,\ldots,x_d) = C^{(d)}(F_d(x_d), C^{(d-1)}(F_{d-1}(x_{d-1}),\ldots)),
$$

where marginals are ordered in an arbitrary way. For example, $C^{(3)}$ can be formed as $C^{(3)}\left(F_1, C^{(2)}(F_2,F_3)\right)$, or as $C^{(3)}\left(F_2, C^{(2)}(F_1,F_3)\right)$, etc. Since no single ordering is preferred we apply model averaging to combine them. This is a central question in the literature on combining multiple prediction densities (Geweke and Amisano, 2011), where optimal weights, also known as scoring rules, are worked out in the context of information theory. As an example, define $c_j^{(3)}$ as $c_j^{(3)} \equiv c^{(3)}(F_j, C_k^{(2)})$, where $j,k = 1,2,3, j \neq k$ and $C_k^{(2)} \equiv C^{(2)}(F_k, F_l), l \neq k, l \neq j$. Then, it is possible in principle to obtain the optimal weights ω_j's as solutions to the following problem:

$$
\max_{\omega_l:\sum\omega_j=1} \sum_{\text{sample}} \ln \sum_j \omega_j c_j^{(3)}
$$

[3]Here by *pseudo*-copula we mean a possibly misspecified copula function. The same term is sometimes used in reference to the empirical copula obtained using univariate empirical cdf's.

Such scoring rules make ω_j's a function of $c_j^{(3)}$'s and may be worth pursuing in large samples. However, it has been noted in this literature that, in finite samples, a simple average often performs better due to the error from estimating ω's (Stock and Watson, 2004). Moreover, in our setting, the optimal weights would need to be solved for in each step, imposing a heavy computational burden.

4 Application

Suppose $d = 5$ and we wish to estimate a time-varying distribution of stock returns with non-trivial conditional skewness and kurtosis. Following a conventional approach, one may use the NAGARCH structure of Engle and Ng (1993) for volatility, the skew-t marginal distributions of Azzalini and Capitanio (2003) for the standardized innovations, and the Student-t 5-copula to model dependence, which amounts to estimating 43 parameters in total.[4] Instead, the proposed algorithm operates with only a few parameters in each step: there are six-parameter marginals in step 1; in steps 2 to 5, there are one-parameter copulas if standard bivariate copula families are used (we use the asymmetrized Student-t 2-copula of Khoudraji (1995) for additional flexibility, which has 4 parameters).

On the other hand, our algorithm would run eighty optimization problems instead of one.[5] So in effect, we replace a single highly-parameterized estimation problem with a long sequence of trivial estimations in which the combined number of parameters is even higher. As a result, our algorithm will produce a better fitting likelihood by construction. We note that our method permits a reduction of the number of estimations by following the approach of Engle, Shephard, and Sheppard (2008) and considering *random* pairs, triples, etc., instead of *all* possible pairs, triples, etc.

In our empirical application we use daily stock returns of five DJIA constituents for the period January 3 to December 31, 2007.[6] We report a few estimates and goodness-of-fit tests. The goodness-of-fit tests use the uniformity and independence properties of probability integral transforms under correct specification (Breymann, Dias, and Embrechts, 2003). We ran Kolmogorov-Smirnov tests of uniformity and F-tests of serial uncorrelatedness.

Table 1 reports sample statistics for five log-returns, and Table 2 reports estimates of the Skew-t-NAGARCH marginals. Table 3 reports the estimates of conventional ML, feasible for $d = 5$. Tables 4-5 contain selected estimates from our algorithm. A few goodness-of-fit statistics are reported

[4]Each skew-t marginal has 6 parameters and the t-copula has 13 distinct parameters.

[5]There are 5 distributions in step 1, 20 distributions in step 2, 30 combinations $\{\widehat{F}_i, \widehat{F}_{jk}\}$ in step 3, 20 combinations $\{\widehat{F}_i, \widehat{F}_{jkl}\}$ in step 4, and 5 combinations $\{\widehat{F}_i, \widehat{F}_{jklm}\}$.

[6]A Matlab module handling arbitrary dimension and data sets under both conventional and sequential methodology is available at `https://sites.google.com/site/artembprokhorov/papers/reconstruct.zip`

in Table 6. The coefficient estimates capture significant degree of skewness and fat tails in the univariate distributions as well as asymmetries in bivariate copulas connecting univariate marginals with marginals of higher dimension. The goodness-of-fit tests show exceptional performance of the new procedure.

5 Conclusion

The sequential ML procedure we propose reconstructs a joint distribution by sequentially applying a copula-like compounding function to lower-dimensional marginal components of the distribution.[7] Clearly, the consistency of this approach hinges on the correctly specified score functions for the lower-dimensional marginals. The sequential structure permits flexible specifications of the score function in each step and more degrees of freedom to come up with a correct specification than using an ‘off-the-shelf’ multivariate distribution.

The issues with conventional ML are not only computational (Hessian non-invertibility, local maxima, etc.). It is often a problem to find a multivariate distributions that accommodates certain features, e.g., asymmetry and extreme dependence in higher dimensions, while remaining tractable. Moreover, finite sample based ML estimation of highly parameterized multivariate distribution is inaccurate due to the curse of dimensionality.

The proposed method falls short of solving all the issues. For example, the full version of the algorithm requires more computing time than conventional ML (when it works) and the standard errors of the sequential procedure suffer from the ‘generated regressor’ problem (Zhao and Joe, 2005). However, the new method allows to estimate distributions with arbitrary patterns of dependence and to parameterize dependence between a scalar and a subvector.

The standard way to study the performance of our algorithm relative to the vine copula and factor copula approaches mentioned in the Introduction is by means of simulations. However, it is unclear what criterion to use for such comparisons. The difficulty is not only in coming up with a feasible version of a MISE-type distance for a d-variate function. The operational version of this measure would need to be applicable to sequential estimators. We leave the development and implementation of such criteria for future research.

[7]This gives the algorithm the flavor of composite likelihood methods (Cox and Reid, 2004).

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	GE	MCD	MSFT	KO	PG
minimum	-0.0384	-0.0298	-0.0421	-0.0285	-0.0506
maximum	0.0364	0.0589	0.0907	0.0254	0.0359
mean, $\times 10^{-3}$	0.0248	1.2831	0.7575	1.0363	0.5987
standard deviation	0.0115	0.0116	0.0143	0.0087	0.0091
skewness	-0.0349	0.2617	0.9461	0.0512	-0.6106
kurtosis	3.9742	4.8977	8.7270	3.6313	9.2954

Table 1: Sample statistics for GE, MCD, MSFT, KO, and PG returns; daily data are from January 3 to December 31, 2007

	GE	MCD	MSFT	KO	PG
μ, $\times 10^{-3}$	-0.032	1.340	0.660	0.750	0.574
	(0.615)	(0.709)	(0.886)	(0.673)	(0.645)
ω, $\times 10^{-5}$	0.569	5.845	0.852	0.667	0.608
	(0.581)	(1.893)	(0.809)	(0.740)	(0.523)
α	0.106	0.153	0.041	0.142	0.107
	(0.062)	(0.090)	(0.024)	(0.082)	(0.052)
β	0.861	0.379	0.915	0.787	0.837
	(0.084)	(0.130)	(0.055)	(0.139)	(0.063)
κ	-0.074	-0.530	-0.174	-0.032	-0.168
	(0.364)	(0.394)	(0.796)	(0.740)	(0.606)
ν	6.482	9.672	5.898	8.098	3.305
	(2.325)	(4.976)	(2.360)	(4.046)	(0.734)
γ	-0.014	-0.431	0.176	-0.236	-0.106
	(0.077)	(0.706)	(0.533)	(0.950)	(0.482)

Table 2: Maximum likelihood estimates of Skew-t-NAGARCH(1,1) marginals: $y_t = \mu + \sqrt{h_t}\varepsilon_t$, $\varepsilon_t \sim$ i.i.d. Skew-t(γ, ν), $h_t = \omega_i + \alpha(y_{t-1} - \mu + \kappa\sqrt{h_{t-1}})^2 + \beta h_{t-1}$

η	13.426
	(4.380)
a	0.030
	(0.035)
b	0.157
	(0.308)

R	GE	MCD	MSFT	KO	PG
GE	1.000	0.425	0.621	0.502	0.510
	(0.000)	(0.055)	(0.042)	(0.055)	(0.049)
MCD	0.425	1.000	0.415	0.398	0.367
	(0.055)	(0.000)	(0.056)	(0.055)	(0.062)
MSFT	0.621	0.415	1.000	0.539	0.465
	(0.042)	(0.056)	(0.000)	(0.053)	(0.057)
KO	0.502	0.398	0.539	1.000	0.495
	(0.055)	(0.055)	(0.053)	(0.000)	(0.049)
PG	0.510	0.367	0.465	0.495	1.000
	(0.049)	(0.062)	(0.057)	(0.049)	(0.000)

Table 3: Maximum likelihood estimates of the time-varying five-dimensional t-copula $C_{\eta,R_t}(u_{1t},\dots,u_{5t})$, where $u_{it} = F^{\mathrm{St}}_{\gamma_i,\nu_i}(\varepsilon_{it})$, with Skew-t-NAGARCH(1,1) marginals: $y_{it} = \mu_i + \sqrt{h_{it}}\varepsilon_{it}$, $\varepsilon_{it} \sim$ i.i.d. Skew-t(γ,ν), $h_{it} = \omega_i + \alpha_i(y_{i,t-1} - \mu_i + \kappa_i\sqrt{h_{i,t-1}})^2 + \beta_i h_{i,t-1}$, and with $R_t = (1-a-b)\overline{R} + a\Psi_{t-1} + bR_{t-1}$, $\Psi_{ij,t-1} = \frac{\sum_{h=1}^m T_\eta^{-1}(u_{it-h})T_\eta^{-1}(u_{jt-h})}{\sqrt{\sum_{h=1}^m T_\eta^{-1}(u_{it-h})^2 \sum_{h=1}^m T_\eta^{-1}(u_{jt-h})^2}}$

	GE,MCD	GE,MSFT	GE,KO	GE,PG	MCD,MSFT
η	9.627	7.948	6.107	14.236	9.883
	(7.732)	(3.006)	(2.390)	(11.290)	(8.488)
a	0.074	0.089	0.002	0.038	0.159
	(0.075)	(0.113)	(0.002)	(0.023)	(0.096)
b	0.399	0.001	0.486	0.913	0.385
	(0.241)	(0.130)	(0.306)	(0.031)	(0.226)
$\bar{\rho}$	0.418	0.625	0.513	0.557	0.429
	(0.062)	(0.042)	(0.050)	(0.076)	(0.073)

Table 4: Maximum likelihood estimates of selected pairwise t-copulas $C_{\eta,\rho_t}(u_{1t},u_{2t})$, where $u_{it} = F^{\mathrm{St}}_{\gamma_i,\nu_i}(\varepsilon_{it})$, with Skew-t-NAGARCH(1,1) marginals: $y_{it} = \mu_i + \sqrt{h_{it}}\varepsilon_{it}$, $\varepsilon_{it} \sim$ i.i.d. Skew-t(γ,ν), $h_{it} = \omega_i + \alpha_i(y_{i,t-1} - \mu_i + \kappa_i\sqrt{h_{i,t-1}})^2 + \beta_i h_{i,t-1}$, and with $\rho_t = (1-a-b)\overline{\rho} + a\psi_{t-1} + b\rho_{t-1}$, $\psi_{t-1} = \frac{\sum_{h=1}^m T_\eta^{-1}(u_{1t-h})T_\eta^{-1}(u_{2t-h})}{\sqrt{\sum_{h=1}^m T_\eta^{-1}(u_{1t-h})^2 \sum_{h=1}^m T_\eta^{-1}(u_{2t-h})^2}}$

$\widetilde{C}(\cdot;\cdot)$	α	β	η	$\overline{\rho}$	a	b
(GE; MCD, MSFT)	0.044	0.322	8.612	0.884	0.004	0.992
(MCD; GE, MSFT)	0.007	0.011	8.557	0.408	0.124	0.351
(MSFT; GE, MCD)	0.008	0.001	8.828	0.503	0.143	0.092
(GE; MCD, MSFT, KO)	0.057	0.326	8.580	0.630	0.007	0.646
(MCD; GE, MSFT, KO)	0.005	0.180	8.326	0.440	0.137	0.244
(MSFT; GE, MCD, KO)	0.022	0.110	8.443	0.508	0.058	0.238
(KO; GE, MCD, MSFT)	0.015	0.062	8.540	0.466	0.023	0.477
(MSFT; GE,MCD,KO,PG)	0.394	0.301	8.580	0.463	0.309	0.285
(KO; GE,MCD,MSFT,PG)	0.285	0.585	8.680	0.433	0.312	0.321
(PG; GE,MCD,MSFT,KO)	0.045	0.252	8.499	0.520	0.051	0.782

Table 5: Maximum likelihood estimates for selected triplets, quadruples and quintuples, with compounding function $C^m(u,v) = u^\alpha v^\beta C_{\eta,\rho_t}(u^{1-\alpha}, v^{1-\beta})$, where $C_{\eta,\rho_t}(\cdot,\cdot)$ is described in Table 4

KS: uniformity	P-value	KS: uniformity	P-value
GE	0.819	MCD|GE	0.635
MCD	0.677	MSFT|MCD	0.833
MSFT	0.972	KO|MSFT	0.997
KO	0.969	PG|KO	0.995
PG	0.851	GE|PG	0.766
F: uncorrelatedness	**P-value**	**F: uncorrelatedness**	**P-value**
GE	0.701	GE: higher moments	0.763
MCD	0.454	MCD: higher moments	0.672
MSFT	0.762	MSF: higher moments	0.635
KO	0.336	KO: higher moments	0.611
PG	0.310	PG: higher moments	0.657

Table 6: Selected goodness-of-fit tests