Online Appendix

Sequential testing with uniformly distributed size

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1 Approximation of baseline boundary for Wiener process

1.1 0^{th} order approximation

We start with the case of one sided testing. The integral equation is

$$\frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{b(r)^2}{2r}\right) = \alpha \int_0^r \frac{1}{\sqrt{2\pi(r-s)}} \exp\left(-\frac{(b(r)-b(s))^2}{2(r-s)}\right) ds,$$

or, after some rearrangement,

$$\int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)-b(s))^2}{2(r-s)}\right) ds = \frac{1}{\alpha}. \tag{1}$$

Let us inspect the integral. Under the integral sign we have a product of two functions of s: $\sqrt{\frac{r}{r-s}}$ and $\exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)-b(s))^2}{2(r-s)}\right)$. The former function is increasing in s and tends to $+\infty$ as s approaches r. The latter function is a bounded continuous function taking values in the interval $\left[0, \exp\left(\frac{b(r)^2}{2r}\right)\right]$ and reaches its maximum value of $\exp\left(\frac{b(r)^2}{2r}\right)$ at $s = r$.

Given the features of these functions we can obtain the *0^{th} order approximation* $b^0(r)$ by replacing $\exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)-b(s))^2}{2(r-s)}\right)$ with its maximum value $\exp\left(\frac{b(r)^2}{2r}\right)$. The 'approximate' integral equation becomes

$$\int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{(b^0(r))^2}{2r}\right) ds = 2r \exp\left(\frac{(b^0(r))^2}{2r}\right) = \frac{1}{\alpha},$$

which gives an explicit solution for the 0^{th} order approximation

$$b^0(r) = \sqrt{2r(-\ln(2\alpha r))}. \tag{2}$$

Note that $b^0(r)$ is a monotonic function in the vicinity of zero.

1.2 1^{st} order approximation

Now we refine the 0^{th} order approximation. Rewrite the integral equation (1) in the form

$$\int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b(r)^2}{2r}\right) \exp\left(-\frac{(b(r)-b(s))^2}{2(r-s)}\right) ds = \frac{1}{\alpha}.$$

Here, three functions get multiplied inside the integral sign. To get the 0^{th} order function we in fact replaced $\exp\left(-\frac{(b(r)-b(s))^2}{2(r-s)}\right)$ by 1. To get the 1^{st} order approximation we replace this function by a piecewise linear approximation. Because

$$\frac{\partial}{\partial s} \exp\left(-\frac{(b(r)-b(s))^2}{2(r-s)}\right)\bigg|_{s=r} = \frac{1}{2}\left(b'(r)\right)^2,$$

we approximate $\exp\left(-\frac{(b(r)-b(s))^2}{2(r-s)}\right)$ by $1-\frac{1}{2}\left(b'(r)\right)^2(r-s)$ in the vicinity of r. Next, we make a guess that for a value of r close to zero the derivative of the boundary $b'(r)$ takes extremely large values so that $\frac{1}{2}\left(b'(r)\right)^2 r > 1$. This means that a linear approximation takes negative values around the zero value of r. As the function to be replaced never takes negative values we make a final adjustment by replacing it by the function $\max\left\{0, 1-\frac{1}{2}\left(b'(r)\right)^2(r-s)\right\}$. The 'approximate' integral equation for the 1st order approximation $b_{\downarrow 0}(r)$ is

$$\int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b_{\downarrow 0}(r)^2}{2r}\right) \max\left\{0, 1-\frac{1}{2}\left(b'_{\downarrow 0}(r)\right)^2(r-s)\right\} ds = \frac{1}{\alpha}.$$

Integration leads to the ordinary differential equation

$$\frac{\varrho\sqrt{r}}{2b'_{\downarrow 0}(r)} \exp\left(\frac{b_{\downarrow 0}(r)^2}{2r}\right) = \frac{1}{\alpha},$$

where

$$\varrho \equiv \frac{8\sqrt{2}}{3},$$

or

$$b'_{\downarrow 0}(r) = \frac{\varrho}{2}\alpha\sqrt{r} \exp\left(\frac{b_{\downarrow 0}(r)^2}{2r}\right), \tag{3}$$

with the initial condition $b_{\downarrow 0}(0) = 0$.

1.3 Asymptotic behavior of 1st order approximation

Let us derive the asymptotic behavior of the 1st order approximation $b_{\downarrow 0}(r)$ in the vicinity of zero. Introduce the new function $z(r)$ related to $b_{\downarrow 0}(r)$ by

$$b_{\downarrow 0}(r) = \sqrt{r}z(r).$$

Having substituted the above expression into (3) we obtain

$$\frac{z(r)}{2\sqrt{r}} + \sqrt{r}z'(r) = \frac{\varrho}{2}\alpha\sqrt{r} \exp\left(\frac{z(r)^2}{2}\right).$$

Next, we guess that asymptotically the first term on the left side is much larger than the second term; call it Claim A. Then we drop the second term on the left side to get an asymptotic solution (without using any new notation)

$$\frac{z(r)}{2\sqrt{r}} = \frac{\varrho}{2}\alpha\sqrt{r} \exp\left(\frac{z(r)^2}{2}\right),$$

or

$$r = \frac{1}{\varrho\alpha} z(r) \exp\left(-\frac{z(r)^2}{2}\right). \tag{4}$$

This is an implicit solution for the asymptotic form of $z(r)$. Based on this solution we can verify

Claim A (see Appendix A). Notice that in the vicinity of 0 the function $z(r)$ is decreasing and $z(0) = +\infty$.

1.4 Asymptotic solution for 1^{st} order approximation

Recall that the PDF of the standard normal distribution is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

The derivative of the PDF is

$$\phi'(x) = -\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right),$$

which implies

$$x \exp\left(-\frac{x^2}{2}\right) = -\sqrt{2\pi}\phi'(x). \tag{5}$$

Combining (4) and (5), we get

$$r = \frac{1}{\varrho\alpha}\left\{-\sqrt{2\pi}\phi'(z(r))\right\},$$

or

$$z(r) = (\phi')^{-1}\left(-\frac{\varrho\alpha}{\sqrt{2\pi}} r\right),$$

where $(\circ)^{-1}$ denotes an inverse function. Recalling that $b_{\downarrow 0}(r) = \sqrt{r}z(r)$ we have the final asymptotic expression for the 1^{st} order approximation:

$$b_{\downarrow 0}(r) = \sqrt{r}(\phi')^{-1}\left(-\frac{\varrho\alpha}{\sqrt{2\pi}} r\right). \tag{6}$$

1.5 Numerical solution for $z(r)$

In implementation, we will need a numerical algorithm of evaluation of $z(r)$ in (4). That is, we need to find the solution of the equation

$$r = \frac{1}{\varrho\alpha} z \exp\left(-\frac{z^2}{2}\right). \tag{7}$$

We do it by the following iterative procedure.

We take a natural logarithm from both sides to get

$$\ln(z) - \frac{z^2}{2} = \ln(\varrho\alpha r),$$

or

$$z = \sqrt{2}\sqrt{\ln(z) - \ln(\varrho\alpha r)}.$$

We take an initial guess $z^0 = \sqrt{2}\sqrt{-\ln(\varrho\alpha r)}$, and then iterate according to

$$z^{k+1} = \sqrt{2}\sqrt{\ln(z^k) - \ln(\varrho\alpha r)}$$

until the sequence z^k converges.

1.6 Calibration of 1st order approximation

Recall that the true baseline boundary obeys equation (1). To understand the features of the asymptotic 1st order approximation $b_{\downarrow 0}(r)$ given by (6), we replace $b(r)$ by $b_{\downarrow 0}(r)$ in (1) and calculate the integral for different values of r. We find that for the value $\alpha \approx 0.131094$ in (6) the right side of (1) is approximately equal to 10 (corresponding to $\alpha = 0.1$) for small values of r.

1.7 Parameterization of baseline boundary

We parameterize the baseline boundary $\Psi(r)$ by the following functional form:

$$\Psi(r) = b_{\downarrow 0}(r) \exp\left(\sum_{j=0}^{J} \psi_j r^j\right).$$

Next recall that in a close vicinity of zero, $b_{\downarrow 0}(r) \approx b(r)$, hence we can safely put $\psi_0 = 0$. Thus, our parametric approximation has the form

$$\Psi(r) = b_{\downarrow 0}(r) \exp\left(\sum_{j=1}^{J} \psi_j r^j\right). \tag{8}$$

We estimate the parameters by OLS in a regression on r, ..., r^J of the difference between the log of the true 10% boundary and the log of $b_{\downarrow 0}(r)$, on a uniform grid of $100,000$ points for $r \in [0.00001, 1.00000]$. The parameters are tabulated in the main text.

1.8 Approximation of two sided baseline boundary

In the case of two sided testing, the integral equation is

$$\begin{aligned}\frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{b(r)^2}{2r}\right) &= \frac{1}{2}\alpha \int_0^r \frac{1}{\sqrt{2\pi(r-s)}} \exp\left(-\frac{(b(r)-b(s))^2}{2(r-s)}\right) ds \\ &\quad + \frac{1}{2}\alpha \int_0^r \frac{1}{\sqrt{2\pi(r-s)}} \exp\left(-\frac{(b(r)+b(s))^2}{2(r-s)}\right) ds.\end{aligned}$$

The counterpart of the integral equation (1) is

$$\int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)-b(s))^2}{2(r-s)}\right) ds + \int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)+b(s))^2}{2(r-s)}\right) ds = \frac{2}{\alpha}.$$

Both integrals take only positive values. Next we evaluate the second integral from above by

$$0 < \int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)+b(s))^2}{2(r-s)}\right) ds \leqslant \int_0^r \sqrt{\frac{r}{r-s}} ds = 2r,$$

where we use that $\frac{b(r)^2}{2r} - \frac{(b(r)+b(s))^2}{2(r-s)} \leqslant 0$. Since $\lim_{r\downarrow 0} 2r = 0$, the asymptotic solution in the vicinity of zero is the solution of the following 'approximate' integral equation

$$\int_0^r \sqrt{\frac{r}{r-s}} \exp\left(\frac{b(r)^2}{2r} - \frac{(b(r)-b(s))^2}{2(r-s)}\right) ds \approx \frac{1}{\alpha/2}. \tag{9}$$

The above equation is the same as the 'one sided' integral equation (1) but with α replaced by $\alpha/2$. Hence, for the two sided boundary we can derive a parametric approximation in exactly the same manner, keeping in mind that we need to 'halve' the total size α.

First, we construct a 1^{st} order boundary approximation. We replace $b(r)$ by $b_{\downarrow 0}(r)$ in (9) and calculate the integral for different values of r. We find that for the value $\alpha \approx 0.130055$ in (9) the right side of (9) is approximately equal to 20 (corresponding to $\alpha = 0.1$) for small values of r.

Similarly to the case of one sided boundary, we parameterize two sided boundary by the same functional form. The parameters are tabulated in the main text.

2 Approximation of baseline boundary for Brownian bridge

2.1 Approximation near zero

Recall that the Brownian bridge is related to the Brownian motion via $B(r) = W(r) - rW(1)$. For values of r close to 0 $B(r)$ is close to $W(r)$, hence we can use the 1^{st} order approximation for values of the boundary near zero.

2.2 Approximation near unity

Surprisingly we get the same 1^{st} order approximation as in the case of the Brownian motion but with a different value of the scaling parameter. Let us derive of the 1^{st} order approximation near $r = 1$ from the integral equation

$$\frac{1}{\sqrt{r(1-r)}} \exp\left(-\frac{b(r)^2}{2r(1-r)}\right) = \alpha \int_0^r \sqrt{\frac{1-s}{(r-s)(1-r)}} \exp\left(-\frac{((1-s)b(r)-(1-r)b(s))^2}{2(r-s)(1-r)(1-s)}\right) ds.$$

After rearrangements we have

$$\int_0^r \sqrt{\frac{(1-s)r}{r-s}} \exp\left(\frac{b(r)^2}{2r(1-r)} - \frac{((1-s)b(r)-(1-r)b(s))^2}{2(r-s)(1-r)(1-s)}\right) ds = \frac{1}{\alpha}. \tag{10}$$

Consider some value of s which is smaller than r and very close to r, so let $r - s = ds$. Then the expression $(1-s)b(r)-(1-r)b(s)$ under the integral sign is approximately equal to

$$\begin{aligned}(1-s)b(r)-(1-r)b(s) &\approx (1-r+ds)b(r)-(1-r)(b(r)-b'(r)ds)\\ &\approx (b(r)+(1-r)b'(r))ds\\ &\approx -\frac{b''(r)}{2}(1-r)^2ds,\end{aligned}$$

where we use the second order approximation that for small values of $(1-r)$:

$$b(r)+(1-r)b'(r)+\frac{b''(r)}{2}(1-r)^2 \approx b(1)=0.$$

Next,

$$\frac{((1-s)b(r)-(1-r)b(s))^2}{2(r-s)(1-r)(1-s)} \approx \frac{\left(-\frac{1}{2}b''(r)(1-r)^2ds\right)^2}{2(1-r)^2ds} = \frac{b''(r)^2}{8}(1-r)^2ds. \tag{11}$$

Given the approximation (11) we can replace (10) with the following ‘approximate' integral equation where we have a different power for the exponent inside the integral:

$$\exp\left(\frac{b(r)^2}{2r(1-r)}\right)\int_0^r \sqrt{\frac{(1-s)r}{r-s}}\exp\left(-\frac{b''(r)^2}{8}(1-r)^2(r-s)\right)ds=\frac{1}{\alpha}.$$

Following the same strategy as in the case of Brownian motion we can approximate the above integral equation as

$$\exp\left(\frac{b(r)^2}{2r(1-r)}\right)\int_0^r \sqrt{\frac{(1-s)r}{r-s}}\max\left\{0,1-\frac{1}{8}\left(b''(r)\right)^2(1-r)^2(r-s)\right\}ds=\frac{1}{\alpha}.$$

This integral equation has a very complicated solution. To overcome this problem we approximate $(1-s)$ with $(1-r)$ inside the square root. As a result we get an approximate solution $b_{\uparrow 1}(r)$ from

$$-\frac{\varrho}{b''_{\uparrow 1}(r)}\sqrt{\frac{r}{1-r}}\exp\left(\frac{b_{\uparrow 1}(r)^2}{2r(1-r)}\right)=\frac{1}{\alpha},$$

or

$$-b''_{\uparrow 1}(r)=\varrho\alpha\sqrt{\frac{r}{1-r}}\exp\left(\frac{b_{\uparrow 1}(r)^2}{2r(1-r)}\right), \tag{12}$$

with the initial condition $b_{\uparrow 1}(1)=0$.

As in the case of Brownian motion, we introduce the function $z(r)$ such that

$$b_{\uparrow 1}(r)=\sqrt{r(1-r)}z(r).$$

Upon substitution into (12) and multiplication of both sides of (12) by $\sqrt{1-r}$ we get

$$\frac{1}{4r^{3/2}(1-r)}z(r)-\frac{1-2r}{\sqrt{r}}z'(r)-r^{1/2}(1-r)z''(r)=\varrho\alpha\sqrt{r}\exp\left(\frac{z(r)^2}{2}\right).$$

Next we need to find the leading term on the left side. As before we make Claim B: on the left side the first term is leading, and prove that the second term and the third term are not dominant. By assuming that asymptotically $r \approx 0$, we have the following asymptotic relation:

$$\frac{1}{4r^{3/2}(1-r)} z(r) = \varrho\alpha\sqrt{r} \exp\left(\frac{z(r)^2}{2}\right),$$

or

$$1 - r = \frac{1}{4\varrho\alpha} z(r) \exp\left(-\frac{z(r)^2}{2}\right). \tag{13}$$

The equation (13) is very similar to the relation (4) except for the multiplier $\frac{1}{4}$. Based on the implicit solution we can verify Claim B (see Appendix A). Note that in the vicinity of 1 the function $z(r)$ is increasing, and $z(1) = +\infty$.

Before moving to the parametrization based on the equation (13) we deal with the counterpart of $z(r)$, which we denote by $\tilde{z}(r, \varkappa)$, where $\varkappa$ is a parameter, this function being implicitly given by the equation

$$r = \varkappa \tilde{z}(r, \varkappa) \exp\left(-\frac{\tilde{z}(r, \varkappa)^2}{2}\right) \tag{14}$$

based on (4). As a result we define

$$\tilde{b}(r, \varkappa) = \sqrt{r(1-r)}\tilde{z}(r, \varkappa). \tag{15}$$

2.3 Numerical solution for $\tilde{z}(r, \varkappa)$

Here we essentially repeat finding the numerical solution for the function $z(r)$. In numerical implementation we need to find a solution of the equation[1]

$$r = \varkappa z \exp\left(-\frac{z^2}{2}\right),$$

or

$$z = \sqrt{2}\sqrt{\ln(z) - \ln\left(\frac{r}{\varkappa}\right)}.$$

We take an initial guess $z^0 = \sqrt{-2\ln(r/\varkappa)}$, and then iterate according to

$$z^{k+1} = \sqrt{2}\sqrt{\ln(z^k) - \ln\left(\frac{r}{\varkappa}\right)}$$

until the sequence z^k converges.

[1] Recall that this equation has a solution for values $r/\varkappa < e^{-1/2} \approx 0.6$.

2.4 Parametrization of boundary on $[0, 1]$

We parameterize the boundary $b_\alpha^R(r)$ by the following functional form:

$$b_\alpha^R(r) = \sqrt{r(1-r)}\tilde{z}(r, \varphi/\alpha)\tilde{z}(1-r, \phi/\alpha)\exp\left(\sum_{j=0}^{J}\psi_j(\alpha)\, r^j\right),$$

where

$$\psi_j(\alpha) = \psi_j^{(0)} + \psi_j^{(1)}\alpha + \psi_j^{(2)}\alpha^2 + \psi_j^{(3)}\alpha^3.$$

Note that the parameters φ and ϕ are different.

Now we describe parameter evaluation in this parameterization. We calibrate the parameter φ by matching boundary values for a small value of r and some value of α to the factor $\sqrt{r(1-r)}\tilde{z}(r, \varphi/\alpha)$ of the parametrization of $b_\alpha^R(r)$.[2] The parameter ϕ is calibrated by matching boundary values for a value of r close to 1 and some value of α to the factor $\sqrt{r(1-r)}\tilde{z}(1-r, \phi/\alpha)$. Next, we obtain the parameter values inside the exponent by running an 'OLS regression' where the dependent variable is a difference between the log of the boundary and the log of $\sqrt{r(1-r)}\tilde{z}(r, \varphi/\alpha)\tilde{z}(1-r, \phi/\alpha)$, on a uniform grid of size 10001×91 for $(r, \alpha) \in [0, 1] \times [0.01, 0.10]$.[3]

Recall that the function $\tilde{z}(r, \varkappa)$ defined by the equation (14), or

$$\frac{r}{\varkappa} = \tilde{z}\exp\left(-\frac{\tilde{z}^2}{2}\right), \tag{16}$$

is defined for those values of $r/\varkappa$ that are lower than $e^{-1/2} \approx 0.6$. To implement a parameterization we need to extend this function for larger values of $r/\varkappa$. The 'extended' function denoted by $\bar{z}(r, \varkappa)$ has the same asymptotic properties but is defined for a larger set of values of the argument. Let $\bar{z}(r, \varkappa)$ be a solution to the following equation for some $c > 0$:

$$\frac{r}{\varkappa} = \bar{z}\exp\left(-\frac{\bar{z}^2}{2}\right)\left(1 + c\exp\left(-\frac{\bar{z}^2}{2}\right)\right).$$

The function $\bar{z}(r, \varkappa)$ is defined for values of $r/\varkappa$ within the interval $\left[0, e^{-1/2}\left(1 + ce^{-1/2}\right)\right]$.[4] This function possesses all necessary properties listed and proved in Appendix B. For the purpose of parameter estimation we set $c = 1$.

The parameters φ and ϕ are calibrated to take values 0.2052 and 0.1433, respectively, for one-

[2] More precisely, for some r and α we solve the following system of equations with two unknowns φ and $\tilde{z}$: $b_{r,\alpha} = \sqrt{r(1-r)}\tilde{z}$, $r = \frac{\varphi}{\alpha}\tilde{z}\exp\left(-\frac{\tilde{z}^2}{2}\right)$, where $b_{r,\alpha}$ is a value of the true boundary. The solution for the parameter φ is

$$\varphi = \frac{\alpha r}{\tilde{z}}\exp\left(\frac{\tilde{z}^2}{2}\right) = \frac{\alpha r\sqrt{r(1-r)}}{b_{r,\alpha}}\exp\left(\frac{b_{r,\alpha}^2}{2r(1-r)}\right).$$

In implementation, we choose $r = 0.0001$ and $\alpha = 0.1$.

[3] The maximal difference in logs between the true and parameterized boundaries is 0.036 for one sided testing and 0.022 for two sided testing.

[4] The actual range is wider but for our purpose it is not important to know it.

sided testing, and 0.4085 and 0.2965 for two sided testing. The other parameters are tabulated in the main text.

2.5 Parametrization of boundary on $[1, +\infty]$

As before, $B(r)$ behaves as $W(r)$ for values of r close to 1, hence we can use the previous 1$^{\text{st}}$ order approximation for values of the boundary near 1, and can use an analogous parameterization, though with respect to the crossing intensity.

For any fixed crossing intensity γ, we use the functional form

$$b_\alpha^M(r) = \sqrt{r-1}\bar{z}(r-1, \varphi/\gamma) \exp\left(\sum_{j=0}^{J} \psi_j(\gamma)(r-1)^j\right),$$

where

$$\psi_j(\gamma) = \psi_j^{(0)} + \psi_j^{(1)}\gamma + \psi_j^{(2)}\gamma^2 + \psi_j^{(3)}\gamma^3.$$

The parameter φ is calibrated by matching boundary values for a value of $r = 1.0001$ and a value of α close to $\gamma = 0.100$ to the factor $\sqrt{r-1}\bar{z}(r-1, \varphi/\gamma)$. We obtain the parameter values inside the exponent by running an 'OLS regression' where the dependent variable is a difference of the log of the boundary and the log of $\sqrt{r-1}\bar{z}(r-1, \varphi/\gamma)$, on a uniform grid of size 10001×101 for $(r, \gamma) \in [1, 11] \times [0.001, 0.100]$.[5]

The parameter φ is calibrated to take value 0.2158 for one-sided testing, and 0.4311 for two sided testing. The other parameters are tabulated in the main text.

A Appendix: auxiliary claims

Proof of Claim A: We want to show that

$$\lim_{r\downarrow 0} \frac{rz'(r)}{z(r)} = 0.$$

Let us compute $z'(r)$. We rewrite (4) as

$$r(z) = \frac{1}{\varrho\alpha} z \exp\left(-\frac{z^2}{2}\right).$$

By taking the derivative we get

$$r'(z) = \frac{1}{\varrho\alpha}\left(1 - z^2\right) \exp\left(-\frac{z^2}{2}\right).$$

Next, due the facts that $r \downarrow 0$ implies $z \to +\infty$, and $z'(r) = 1/r'(z)$, it follows that it suffices

[5]The maximal difference in logs between the true and parameterized boundaries is 0.068 for both one sided and two sided testing.

to show that

$$\lim_{z\to+\infty}\frac{r(z)}{zr'(z)}=\lim_{z\to+\infty}\frac{1}{z\left(1-2z^2\right)}=0.$$

□

Proof of Claim B: We have three terms: $\frac{1}{4}r^{-3/2}\left(1-r\right)^{-1}z(r)$, $r^{-1/2}\left(1-2r\right)z'(r)$ and, finally, $r^{1/2}\left(1-r\right)z''(r)$. The leading asymptotic components of these terms are $\left(1-r\right)^{-1}z(r)$, $z'(r)$ and $(1-r)z''(r)$. We want to show that

$$\lim_{r\downarrow 0}\frac{(1-r)z'(r)}{z(r)}=0$$

and

$$\lim_{r\downarrow 0}\frac{(1-r)^2z''(r)}{z(r)}=0.$$

The first limit can be proved exactly in the same fashion as in Claim A.

It is known that

$$z''(r)=-\frac{r''(z)}{r'(z)^3},$$

where $r(z)$ is given by (13), or

$$r=1-\frac{1}{2\varrho\alpha}z\exp\left(-\frac{z^2}{2}\right).$$

Then

$$\begin{aligned}\lim_{r\downarrow 0}\frac{(1-r)^2z''(r)}{z(r)} &= \lim_{z\to+\infty}-\frac{(1-r)^2r''(z)}{zr'(z)^3}\\ &= \lim_{z\to+\infty}\frac{z^2\left(3-z^2\right)}{(1-z^2)^3}=0.\end{aligned}$$

□

B Appendix: properties of $\bar{z}$

Recall that $\bar{z}(r,\varkappa)$ is a solution of the following equation:

$$\frac{r}{\varkappa}=\bar{z}\exp\left(-\frac{\bar{z}^2}{2}\right)\left(1+c\exp\left(-\frac{\bar{z}^2}{2}\right)\right),$$

where $c>0$. Note that the function $\bar{z}(r,\varkappa)$ is in fact a function of the ratio $r/\varkappa$, which we denote by x. Hence we can define a function of one argument $\bar{z}(x)$ by

$$x=\bar{z}\exp\left(-\frac{\bar{z}^2}{2}\right)\left(1+c\exp\left(-\frac{\bar{z}^2}{2}\right)\right). \tag{17}$$

Similarly, we can define the function $\tilde{z}(x)$, which is a solution of (17) when $c = 0$. Below we list and prove a set of properties.

Property 1: The function $\bar{z}(x)$ is well defined for values $x \in \left(0, e^{-1/2}\left(1 + ce^{-1/2}\right)\right]$, and it takes on that interval the values no smaller than unity.
Proof: We take a derivative of the right side of (17) with respect to $\bar{z}$ and obtain

$$\frac{\partial}{\partial \bar{z}} \bar{z} \exp\left(-\frac{\bar{z}^2}{2}\right)\left(1 + c\exp\left(-\frac{\bar{z}^2}{2}\right)\right) = (1 - \bar{z}^2)\exp\left(-\frac{\bar{z}^2}{2}\right) + c(1 - 2\bar{z}^2)\exp\left(-\bar{z}^2\right).$$

This derivative is negative for $\bar{z} \geq 1$. Hence, the right side of (17) is a decreasing function of $\bar{z}$ for $\bar{z} \geq 1$, and it decreases from the value $\left[(1 - \bar{z}^2)\exp\left(-\frac{\bar{z}^2}{2}\right) + c(1 - 2\bar{z}^2)\exp\left(-\bar{z}^2\right)\right]_{\bar{z}=1} = e^{-1/2}\left(1 + ce^{-1/2}\right)$ to 0. □

Property 2: For $x \in \left(0, e^{-1/2}\left(1 + ce^{-1/2}\right)\right]$, the following inequality holds: $\tilde{z}(x) < \bar{z}(x)$.
Proof: For given $\bar{z} \geq 1$ the right side of (17) is a strictly monotone function of c. □

Property 3: In the vicinity of zero, the function $\bar{z}(x)$ has the same asymptotics as $\tilde{z}(x)$, i.e.

$$\lim_{x \downarrow 0} \frac{\bar{z}(x)}{\tilde{z}(x)} = 1.$$

Proof: Given property 2, suppose that the opposite holds, i.e. there exists $\varepsilon > 1$ such that for any $\delta > 0$ there exists $x \leq \delta$ such that

$$\frac{\bar{z}(x)}{\tilde{z}(x)} \geq \varepsilon.$$

For any such x we know that

$$\tilde{z}\exp\left(-\frac{\tilde{z}^2}{2}\right) = \bar{z}\exp\left(-\frac{\bar{z}^2}{2}\right)\left(1 + c\exp\left(-\frac{\bar{z}^2}{2}\right)\right)$$

where the argument x is dropped for the simplicity of exposition. From the statement above we have $\bar{z} \geq \varepsilon\tilde{z}$ (as $\varepsilon > 1$), and $\tilde{z}$ can be sufficiently large. Let us rewrite the above equality as

$$\frac{\bar{z}}{\tilde{z}}\exp\left(-\frac{(\bar{z}/\tilde{z})^2 - 1}{2}\tilde{z}^2\right)\left(1 + c\exp\left(-\frac{\bar{z}^2}{2}\right)\right) = 1.$$

For sufficiently large $\tilde{z}$ the above equality cannot hold. Indeed, for $\tilde{z} \geq 1$ and $\bar{z}/\tilde{z} \geq \varepsilon$, the expression

$$\frac{\bar{z}}{\tilde{z}}\exp\left(-\frac{(\bar{z}/\tilde{z})^2 - 1}{2}\tilde{z}^2\right)$$

reaches its maximum at $\bar{z}/\tilde{z} \geq \varepsilon$, this maximum being equal to $\varepsilon \exp\left(-\frac{\varepsilon^2-1}{2}\tilde{z}^2\right)$, so

$$\frac{\bar{z}}{\tilde{z}} \exp\left(-\frac{(\bar{z}/\tilde{z})^2-1}{2}\tilde{z}^2\right)\left(1+c\exp\left(-\frac{\bar{z}^2}{2}\right)\right) \leq \varepsilon \exp\left(-\frac{\varepsilon^2-1}{2}\tilde{z}^2\right)\left(1+c\exp\left(-\frac{\tilde{z}^2}{2}\right)\right),$$

which is strictly smaller than unity for sufficiently large $\tilde{z}$. The contradiction shows that the statement of the property 3 holds. □

Property 4: The function $\bar{z}(r,\varkappa)$ can be calculated by the following iterative algorithm. From the initial guess $z^0 = \sqrt{2}\sqrt{-\ln(r/\varkappa)}$, iterate according to

$$z^{k+1} = \sqrt{2}\sqrt{\ln(z^k) + \ln\left(1+c\exp\left(-\frac{\bar{z}^2}{2}\right)\right) - \ln\left(\frac{r}{\varkappa}\right)}$$

until the sequence z^k converges.