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On Microfoundations of the Dual Theory of Choice

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Abstract

We show that Yaari's dual theory of choice under risk may be derived as an indirect utility when a risk-neutral agent faces financial imperfections. We consider an agent that maximizes expected discounted cash flows under a bid-ask spread in the credit market. It turns out that the agent evaluates lotteries as if she were maximizing Yaari's dual utility function. We also generalize the dual theory of choice for unbounded lotteries.

Key words: dual theory of choice, financial imperfections

JEL classification No.: D81, G11

1. Introduction

The dual theory of choice under risk (DTC) was introduced in Yaari [1987] in order to resolve certain theoretical problems with expected utility. One of the most important properties of the latter is that risk aversion is equivalent to diminishing marginal utility. Yaari argued that risk aversion and diminishing marginal utility are 'horses of different colors' and put forward a theory which allowed for linear risk averse utilities. The linearity property makes modelling behavior of risk-averse agents much simpler. This is why DTC has become a popular tool for testing robustness of various economic models that have been so far analyzed only in the expected utility framework. Demers and Demers [1990] apply DTC to firms' production decisions. Hadar and Seo [1995] examine portfolio choice and diversification in DTC. Doherty and Eeckhoudt [1995] study optimal insurance, Epstein and Zin [1990] use DTC as the simple utility with first-order risk aversion to offer an explanation of the equity premium. Volij [1999] studies revenue equivalence in auction with bidders that maximize DTC utility. Schmidt [1998] applies DTC to principal-agent problems.

Despite being a handy tool in theoretical models, the dual theory of choice has not done well in experimental studies. Both Harless and Camerer [1994] and Hey and Orme [1994] showed that DTC performs rather badly not only in absolute terms but also relative to other non-expected utility theories as well as relative to the expected utility theory. This may not be surprising since the experiments were carried out on individuals. One should expect individuals to have diminishing marginal utility. On the other hand, firms and banks, especially in the long-run should have linear objective function. In the meanwhile, there exists a substantial empirical evidence that firms and even banks are risk-averse [see Rose, 1989; Davidson et al., 1992; Park and Antonovitz, 1992 etc.].

In the expected utility theory, risk-aversion of firms can be explained by market imperfections [Greenwald and Stiglitz, 1990]. If credit markets are imperfect, a firm prefers certainty: a mean-preserving spread of project payoffs decreases expected profits after interest. Even if the firm is orginally risk-neutral and maximizes expected cash flows, market imperfections may make it risk-averse.

The goal of our paper is to provide similar microfoundations for the dual theory of choice. We also consider a firm that is initially risk neutral. The firm evaluates lotteries (or project portfolios) according to the amount of expected discounted cash flows that can be obtained with those lotteries. We assume that the firm faces bid-ask spread in the financial market, i.e. the interest rate on loans is higher than the interest rate on deposits. The innovation of the paper is to assume that the firm anticipates the future need for borrowing if the returns are low and saving extra funds if the returns are high. Therefore firm can adjust its financial position before realization of stochastic payoffs. We derive indirect utility as a function of the distribution of returns and show that the indirect utility belongs to a certain subset of DTC utilities.

Another contribution of the paper is a generalization of DTC for unbounded random variables. Although infinite payments are not likely to occur in real world, in models they are quite common (e.g. normal distribution). We obtain representation for random variables with finite means. Our representation form turns out to be similar to one introduced in Roell [1987].

The paper is organized as follows. In Section 2 we introduce notation and obtain representation of DTC for unbounded lotteries. In Section 3, we show that an agent that faces a bid-ask spread in the financial market, evaluates lotteries as if she were a DTC utility maximizer. In Section 4 we check whether this result also holds for an arbitrary (not necessarily piecewise constant) interest rate schedule and show that it does not. Section 5 concludes. The Appendix contains proofs.

2. Dual theory of choice revisited

In this Section we extend the dual theory of choice to the case of unbounded random variables with finite means. In addition to original Yaari's axioms, we assume finiteness of certainty equivalents. Then we obtain a representation result, and characterize risk aversion.

2.1. *Notation*

We shall consider a preference relation $\succeq$ over real-valued random variables ('lotteries') X. We consider a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ where Ω is the set of states of nature, $\mathcal{A}$ is a σ-algebra on Ω, $\mathcal{P}$ is a probability measure on $\mathcal{A}$. A random variable is a real-valued function of state of nature $X : \Omega \longmapsto \Re$. For each random variable we introduce a cumulative distribution function $F_X(x) = \text{Prob}\{X \leq x\}$. The distribution functions are right-continuous, non-decreasing and map $[-\infty, +\infty]$ onto $[0, 1]$. We will also use inverse distribution functions (IDF) $\Phi(p) = F^{-1}(p) = \sup\{x : p \in \hat{F}(x)\}$, where $\hat{F}(x) = \{p : F(x-0) \leq p \leq F(x)\}$

is the *set-valued distribution function*. According to this definition, a point (p, x) belongs to the graph of $\Phi(p)$ if and only if the point $(x,\ p)$ belongs to the graph of $F(x)$. $\Phi(p)$ is a non-decreasing function that maps [0, 1] onto $[-\infty, +\infty]$.

We will consider all random variables with finite means so that $|\int_{-\infty}^{+\infty} x dF(x)| = |\int_0^1 \Phi(p)dp| < \infty$. Then $\Phi(p) \in L^1(0, 1)$ and $F(x) \in L^1(-\infty, +\infty)$. Denote M a set of all non-decreasing upper semi-continuous functions that belong to $L^1(0, 1)$.[1] Then M includes IDFs of all random variables with finite means. M is a semi-linear space. Indeed, for any $\Phi_1, \Phi_2 \in M$ and $\alpha \geq 0$ we have $\Phi_1 + \Phi_2 \in M$ and $\alpha\Phi_1 \in M$. There is a zero element in M : $\Phi_0(p) \equiv 0$ for all $p \in [0, 1]$. Indeed, $\Phi_0 = 0 \cdot \Phi$ and $\Phi_0 + \Phi = \Phi$ for any $H \in M$.

We do not require $\Phi(0)$ or $\Phi(1)$ to be finite and therefore allow for unbounded random variables. The L^1 norm is very convenient for dealing with IDFs: the distance between two distribution functions $\int_{-\infty}^{+\infty} |F^1(x) - F^2(x)|dx$ in terms of space $L^1(-\infty, +\infty)$ coincides with distance between corresponding IDFs $\|\Phi^1 - \Phi^2\| = \int_0^1 |\Phi^1(p) - \Phi^2(p)|dp$ in terms of $L^1(0, 1)$.

We shall also use the concept of comonotonicity.

Definition 1: *Two random variables X and Y are said to be comonotonic if* $(X(\omega) - Y(\omega))(X(\omega') - Y(\omega')) \geq 0$ *holds for any pair of states of nature* ω, ω'.

We will denote $C(b)$ a degenerate random variable that takes real value b with probability $1 : \Phi_{C(b)} \equiv b$.

2.2. *Dual theory of choice for unbounded lotteries*

We use Yaari's axioms i.e. the axioms of expected utility theory where the von Neuman-Morgenstern independence axiom is replaced with Yaari's dual independence axiom. Then we introduce an additional condition that allows us to deal with unbounded random variables.

Axiom 1 (Neutrality ($A1$)): If inverse distribution functions of two random variables X and Y coincide (i.e. $\Phi_X(\cdot) = \Phi_Y(\cdot)$) then $X \sim Y$.[2]

This means that instead of dealing with random variables we can simply define preference relation on M. By definition, $\Phi_X(\cdot) = \Phi_Y(\cdot)$ is equivalent to $F_X(\cdot) = F_Y(\cdot)$. Hereinafter, we shall understand $\Phi_X(\cdot) = \Phi_Y(\cdot)$ as equality almost everywhere (i.e. maybe except for a set of measure zero).

Axiom 2 (Complete weak order ($A2$)): Preference $\succeq$ is a complete weak order.

Axiom 3 (Continuity ($A3$)): The preference relation is continuous on M in terms of L^1 norm i.e. if $X \succ Y$ then there exists $\varepsilon > 0$ such that for all Z with finite means $\|\Phi_Z - \Phi_Y\| < \varepsilon$ implies $X \succ Z$.

Axiom 4 (Monotonicity ($A4$)): If $\Phi_X(p) \geq \Phi_Y(p)$ for every $p \in [0, 1]$ then $X \succeq Y$.

The monotonicity axiom A4 is equivalent to conventional monotonicity in terms of first-order stochastic dominance (FOSD). Indeed, $\Phi_X(\cdot) \geq \Phi_Y(\cdot)$ is equivalent to the well-known FOSD condition $F_X(\cdot) \leq F_Y(\cdot)$.

Axiom 5 (Dual independence ($A\,5$)): If random variables X, Y and Z are pairwise comonotonic then for every real number $\alpha \in [0, 1]$, $X \succeq Y$ implies $\alpha X+(1-\alpha)Z \succeq \alpha Y+(1-\alpha)Z$.

The dual independence axiom introduces linearity with regard to *payoffs* like von Neuman-Morgenstern independence axiom makes sure that utility is linear with regard to *probabilities*. We will recurrently refer to this duality between payoffs and probabilities throughout the paper. Notice that A5 imposes linearity only upon the comonotonic lotteries (i.e. ones that cannot be used as a hedge against each other); otherwise the preferences would have to be risk neutral.

Axiom 6 (Finite certainty equivalent ($A\,6$)): For each random variable with finite mean X, there exist such (finite) real values a and A that $C(a) \prec X \prec C(A)$.

The first five axioms are precisely the ones introduced by Yaari [1987]. If the preference were defined on the set of uniformly bounded random variables, axioms A1–A5 would be sufficient to characterize dual theory theory of choice. However, since our goal is to study utilities that can also compare unbounded lotteries, we need to introduce an additional axiom A6 which states that each lottery (with a finite mean) has a finite certainty equivalent. In the original Yaari's setting A6 is automatically satisfied: $a = 0$ and $A = 1$. Notice that A6 does not require a and A to be the same for all lotteries. The axiom A6 is sufficient to extend the dual theory of choice to the case of unbounded lotteries. In order to obtain the representation result, we first describe the certainty equivalent functional.

Proposition 1: *If* $A1$–$A6$ *hold*, *there exists a functional* U *that assigns a real number* (*'certainty equivalent'*) *to any random variable* (*with a finite mean*) *such that* $X \succeq Y$ *if and only if* $U(X) \geq U(Y)$. *For any real a holds* $U(C(a)) = a$. *If random variables* X *and* Y *are comonotonic*, $U(X + Y) = U(X) + U(Y)$, *where* $U(\cdot)$ *is the certainty equivalence functional. Also, for any random variable* X *and any real* $\alpha \geq 0$ $U(\alpha X) = \alpha U(X)$.

All Proofs are provided in the Appendix unless stated otherwise.

Let us also define the certainty equivalent functional for inverse distribution functions.

Definition 2: *Consider a preference relation* $\succeq$ *that satisfies axioms* $A1$–$A6$. *A functional* $V(\cdot)$ *that assigns a real number to any function* $\Phi \in M$ *is said to be a certainty equivalent for inverse distribution functions if for every random variable* X *with finite mean* $V(\Phi_X) = U(X)$.

According to A1, the definition is consistent: if two variables have the same IDF then they are equivalent, hence the value of certainty equivalent is the same. Therefore $X \succeq Y$ is equivalent to $V(\Phi_X) \geq V(\Phi_Y)$.

Lemma 1: *Preference relation satisfies* $A1$–$A6$ *if and only if the corresponding certainty equivalent functional defined on IDFs is a linear continuous functional on* M *i.e. for all inverse distribution functions* $\Phi^1, \Phi^2 \in M$ *and non-negative real* α:

- $V(\alpha\Phi^1) = \alpha V(\Phi^1)$.
- $V(\Phi^1 + \Phi^2) = V(\Phi^1) + V(\Phi^2)$.
- *If* $\Phi^1(p) \equiv 1$ for all p then $V(\Phi^1) = 1$.

Comment. We could have also replaced 'for any non-negative α' with 'for any real α' and the statement would still be the same because there is no negative α such that both Φ^1 and $\alpha\Phi^1$ belong to M (inverse distribution functions must be non-decreasing).

Now we can easily prove the representation result.

Theorem 1: *Preference relation satisfies* A1–A6 *if and only if the corresponding certainty equivalent functional defined on IDFs can be represented in the following form*:

$$V(\Phi) = \int_0^1 h(p)\Phi(p)\,dp \tag{1}$$

where $h(p) \in L^\infty[0, 1]$, $h(p) \geq 0$ *and* $\int_0^1 h(p)\,dp = 1$.

The idea of the proof is to apply a well-known result of functional analysis that the space of linear continuous functionals is isomorphic to the conjugate space.[3] The weight function $h(p)$ must belong to $L^\infty(0, 1)$ because we allow for all random variables with finite means. If we allowed only for random variables with both finite mean and variance we would have $h(p) \in L^2(0, 1)$ instead ($L^2(0, 1)$ is the conjugate space for $L^2(0, 1)$).

What is the economic interpretation of (1)? The representation (1) emphasizes that Yaari utility is a linear rank-dependent utility, i.e. it maximizes a weighted average of payments in different states of nature with weights being a function of the *rank* of each state. Indeed, since $\Phi(p)$ is a monotonic function, the states are ranked in the order of increasing payoffs, and p is precisely the rank of each state. The weight function $h(p)$ shows how much weight is given to the relatively good states and to the relatively bad ones.

Remark: If $F_X(x)$ is continuous then utility may also be represented in terms of cumulative distribution function

$$U(X) = \int_{-\infty}^{+\infty} h(F_X(\xi))\xi\,dF_X(\xi). \tag{2}$$

Note that by construction this representation gives utility which is properly defined for all random variables with finite means including the unbounded ones. In the meantime, Yaari's representation

$$U = \int_{-\infty}^{+\infty} g(1 - F_X(\xi))\,d\xi \tag{3}$$

is defined only for random variables that are uniformly bounded. Indeed, if X is unbounded then

$$\int_{-\infty}^{+\infty} g(1 - F_X(\xi))\,d\xi = \int_{-\infty}^{0} g(1 - F_X(\xi))\,d\xi + \int_{0}^{+\infty} g(1 - F_X(\xi))\,d\xi$$

is finite only if $g(0) = g(1) = 0$. But monotonicity axiom requires that g is non-decreasing. Thus the only functional (3) defined for unbounded random variables is the trivial one $g \equiv 0$.

For uniformly bounded random variables, though, integration by parts converts (2) to Yaari's form and back (which has been done in Roell [1987] and Demers and Demers [1990]. Indeed, suppose that we only consider lotteries with $X \in [0, 1]$ with probability 1. Then

$$\int_0^1 g(1 - F_X(\xi))\, d\xi = g(0) + \int_0^1 g'(1 - F_X(\xi))\xi dF_X(\xi).$$

Taking

$$h(p) = g'(1 - p) \tag{4}$$

we obtain (2).

2.3. *Risk-aversion*

Similarly to Yaari's characterization of risk aversion in DTC [Yaari, 1986, 1987], we shall determine conditions on $h(\cdot)$ for utility (1) to be risk-averse.

In order to define risk aversion, we use Rotschild-Stiglitz concept of mean-preserving spread. Consider arbitrary random variable X (with finite expected value) and an uncorrelated noise $\xi(E(\xi \mid X) = 0)$. Then $X + \xi$ is a mean-preserving spread of X and therefore risk-averse agents should prefer X to $X + \xi$.

Definition 3: *Utility $U(\cdot)$ is said to be risk-averse if $U(X) \geq U(X + \xi)$ for all X and ξ such that $E(\xi \mid X) = 0$.*

It is well known that if neutrality axiom A1 holds, such definition of risk-aversion can be re-written in terms of distribution functions. Utility functional is *risk averse* if for any random variables X and Y with the same mean $EX = EY$ the following is true: if inequality

$$\int_{-\infty}^a F_X(x)\, dx \leq \int_{-\infty}^a F_Y(y)\, dy \tag{5}$$

holds for any real a then $U(X) \leq U(Y)$.[4]

Let us reformulate risk aversion in terms of IDF. Condition (5) is equivalent to

$$\|\Phi_X(p) - a\| \leq \|\Phi_Y(p) - a\|. \tag{6}$$

Theorem 2: *The utility functional* (1) *is risk averse if and only if the weight function $h(p)$ is non-increasing.*

This result is perfectly consistent with Yaari's characterization of risk aversion: the utility is risk averse whenever the distortion function $g(\cdot)$ is convex [Yaari, 1986, 1987]. Since for uniformly bounded random variables the weight function is related to the first derivative of g (see (4)), the convexity of $g(\cdot)$ is equivalent to monotonicity of $h(\cdot)$.

The result is very intuitive. Indeed, take a lottery X and consider a mean preserving spread Y, that pays the same on average, but pays less in bad states and more in good states. The utility functional is risk averse $U(X) \geq U(Y)$ if and only if the loss from getting less in the bad states is greater than the benefit from getting more in the good states. This implies that risk aversion is equivalent to larger weights for bad states and smaller weights for good states. Hence, the weight function $h(p)$ must be non-increasing.

3. Microfoundations of dual theory of choice

In this Section we provide 'microfoundations' for the dual theory of choice. In this paper, microfoundations are understood as a model with rational choice that explains why an agent who is originally risk-neutral may end up having a risk averse dual utility. In the expected utility world, the story may go as follows: consider an agent that maximizes expected income and faces financial market imperfections, i.e. the costs of borrowing are higher than interest rate on savings. In this situation the agent would prefer less risky alternatives. This framework may also explain risk loving preferences. Indeed, consider a situation where the agent is not punished at all for negative income. Under limited liability, firm's owners get zero income if firms' profit is negative but get all the profit whenever it is positive. Then the owners would prefer more risky alternatives: in the good states their return is very high while in the bad state the punishment is bounded from below. In this Section we build a similar model for DTC preferences.

3.1. The basic model

We consider a simple two-period model of a risk-neutral agent that faces a bid-ask spread in the credit market. Namely, the interest rate the agent pays on her loans is R_l while she can only save at the risk-free rate $R_s < R_l$. The agent is neutral to risk and maximizes expected discounted withdrawn earnings in periods 0 and 1 with the discount rate Δ. The agent has constant marginal utility so that the model below is applicable to firms rather than to households.[5]

In order to rule out trivial cases, we assume (all rates are gross)

$$R_s < \Delta < R_l. \tag{7}$$

The agent is to evaluate a lottery (e.g. a risky project) that pays X in the period 1. The distribution function of payoffs $F(\cdot)$ is such that the expected value of X is finite: $|EX| = |\int_{-\infty}^{+\infty} x dF_X(x)| = |\int_0^1 \Phi_X(p)\, dp| < \infty$.

The agent chooses to withdraw C in the first period. If $X > C$ then the agent saves $S = X - C$ and therefore the second-period payoff will be $R_s(X - C)$. If $X < C$ then the agent will have to borrow $L = C - X$ and will have to pay $R_l(C - X)$ in the second period.

The expected present discounted value of the project is[6]

$$
\begin{aligned}
U(X) &= \int_{-\infty}^{+\infty} dF_X(x)\{C + \Delta^{-1}(R_s[x - C]_+ - R_l[C - x]_+)\} \\
&= \int_0^1 dp\{C + \Delta^{-1}(R_s[\Phi_X(p) - C]_+ - R_l[C - \Phi_X(p)]_+)\}
\end{aligned} \tag{8}
$$

It is natural to assume that the agent chooses C to maximize (8). To describe the choice of C, we need to specify, however, whether the agent chooses C knowing the realization of X or at least its distribution. We shall calculate the agent's evaluation of the project $U(X)$ for three scenarios.

First, we will consider the case of extreme flexibility where the first period spending decision C is taken after the realization of X is observed (timeline (a) in figure 1). The opposite extreme is to assume that the agent cannot vary C at all (timeline (b) in figure 1). Suppose that $C = C^*$ is exogenously set by someone else (or by the agent herself but before she even learns $F_X(\cdot)$). The third case is intermediate: the agent can vary C but has to take this decision before X is observed (timeline (c) in figure 1). In this case C cannot depend on X but can depend on the *distribution* of X.

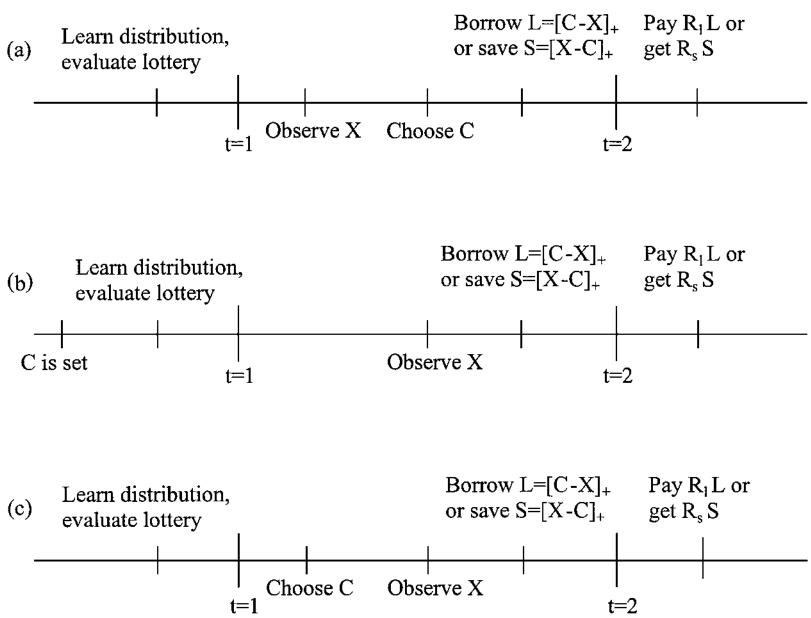

Figure 1. Agent's evaluation of a lottery X depends on timeline. Under timeline (a) the agent chooses first-period withdrawals C *after* observing realization of X. In this case she is risk-neutral and prefers a lottery with higher EX. In the case (b) where the agent evaluates the lottery after C is set, her preferences are described by a risk-averse expected utility. If the agent chooses first-period withdrawals C *before* knowing realization of X (case (c)) she is risk-averse and maximizes a Yaari utility (9).

In the first case (timeline (a)), the solution is straightforward: take $C(X) = X$. Indeed, (7) implies that saving pays off too little, so that the agent prefers consumption to saving; borrowing is too costly, so that borrowing to consume is not rational. Substituting $C(X) = X$ into (8), we obtain $U = EX$—agent remains risk-neutral.

The second case (timeline (b)) is very different. Substituting $C = C^*$ into (8) we obtain an expected utility similar to those discussed in Greenwald and Stiglitz [1990], Eeckhoudt et al. [1997]. Indeed, agent gets expected value of a concave piecewise-linear utility function with a kink at $X = C^*$. The agent maximizes $Eu(X)$ where $u(X) = R_s[X - C^*]_+ - R_l[C^* - X]_+$. Eeckhoudt et al. [1997] analyze such preferences in detail and show that they have first-order risk aversion.

The innovation of our paper is to study the third, intermediate case (timeline (c)) where the choice C depends on distribution but not on realization of X. In this case, the agent's evaluation of the lottery is

$$U(X) = \max_C \left\{ C - \frac{R_l}{\Delta} \int_{-\infty}^{C} (C - x)\, dF_X(x) + \frac{R_s}{\Delta} \int_{C}^{\infty} (x - C)\, dF_X(x) \right\}. \tag{9}$$

Proposition 2: *The solution C to maximization problem* (9) *satisfies*[7]

$$F_X(C - 0) \leq (\Delta - R_s)/(R_l - R_s) \leq F_X(C). \tag{10}$$

If $F_X(\cdot)$ *is continuous*, (10) *takes the form*

$$F_X(C) = (\Delta - R_s)/(R_l - R_s) \tag{11}$$

Proof trivially follows from the first- and second-order conditions.

Formulas (10) and (11) implies $C = \Phi_X(p^*)$, where $p^* = (\Delta - R_s)/(R_l - R_s)$. The first-period spending is simply the value of inverse distribution function of the project's payoff taken at a given point $(\Delta - R_s)/(R_l - R_s)$. Therefore if project's payoff increases by a dollar in all states of nature, C also goes up by one dollar. If payoffs in all states double, C doubles as well. If there are two comonotonic projects, the corresponding values of the first-period consumption add up. Thus, the first period consumption $C = \Phi_X(p^*)$ is a linear functional defined on the lotteries so there is little wonder than overall evaluation of the project $U(\cdot)$ is a Yaari functional.

Proposition 3: *Formula* (9) *defines a DTC utility functional* (1) *with the weight function*

$$h(p) = \begin{cases} R^B & \text{if } p < p^* \\ R^G & \text{if } p \geq p^* \end{cases} \tag{12}$$

(see figure 2), where $R^G = R_s/\Delta$, $R^B = R_s/\Delta$, $p^* = (1 - R^G)/(R^B - R^G)$.

The formula (9) describes the simplest possible risk-averse DTC functional. The agent maximizes a weighted average of payoffs with higher weights R^B for bad states of nature $(p < p^*)$ and lower weights R^G for good states of nature. The only weight function that

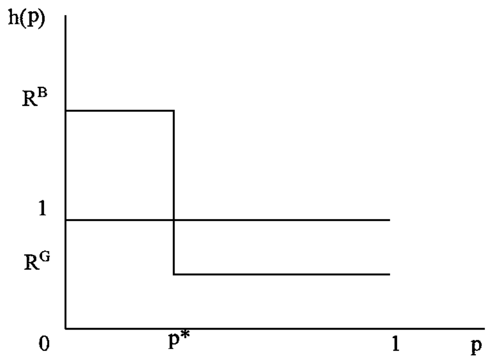

Figure 2. The simplest risk averse DTC utility (12).

is simpler than (12) is the one which takes only one value in all states $h(p) \equiv 1$, but the corresponding utility is risk neutral.

The statement opposite to Proposition 3 is also true. For an arbitrary risk-averse DTC utility functional (1) such that $h(p)$ takes only two values, there exist such R_l and R_s that this functional can be derived as the value function in (9). This class of utilities is the set of simplest possible risk-averse rank-dependent utilities that are defined as follows: the states are ranked in the order of payoffs and divided into the set of the 'good' ones and that of the 'bad' ones. The agent's payoff of getting a dollar in a good state is R^G, while that of getting a dollar in a bad state is $R^B > R^G$. Such utilities are fully characterized by two parameters. The first one is the cutoff p^* between the 'good' states of nature and the 'bad' ones. The other parameter is the relative difference of utility of getting a dollar in a good state and that of getting a dollar in a bad state $r = R^G/R^B$. Using normalization $1 = \int_0^1 h(p)dp = R^B p^* + (1 - p^*)R^G$ we obtain (12).

Let us compare the results for the three scenarios. If the agent chooses C after she learns X, (case (a)) she adjusts C accordingly and remains risk neutral $U = EX$. If she can only adjust C *before* she learns X (case (c)), then she takes a risk that she would need to borrow in some states and save in others. Since financial market is imperfect, she gets less than EX. However, by adjusing C, the agent minimizes the losses due to financial market imperfections given the distribution. If she cannot adjust C at all (case (b)) then the evaluation decreases even further away from the expected value EX. Since $C = C^*$ is fixed, the utility is less than in case (c) (unless C^* luckily coinsides with the optimal choice $\Phi_X(p^*)$ which is not a generic case).

The cases (b) and (c) are *dual* in terms of Yaari's duality between payoffs and probabilities. In both cases, the agent calculates a weighted average of payoffs, the weights are the same but the concept of weighing is different. In both cases the agent assigns high weight R^B to each dollar received in a bad state and low weight R^G to each dollar received in a good state. The difference is in the definition of the good vs. bad states. In the dual utility case (c) the cutoff is based on *probability*: the bad state is defined as the state with $p < p^*$ (and the states with $p > p^*$ are good ones). In the expected utility case (b) the cutoff is bases on *payoffs*: bad states are the ones with $X < C^*$. The two functionals are equal to each other whenever $C^* = \Phi_X(p^*)$ (or $p^* = F_X(C^*)$) but this coincidence occurs only for a very small subset of lottery space.

Besides the capital market imperfections, the analysis above can also be applied to an agent operating under a piece-wise linear tax schedule. Consider an agent that faces, positive tax rate for positive net cash flows and zero tax for negative cash flows ('no-loss-offset' rule). The model would identical to the one above. The discontinuity in the tax rate makes effective R^G lower: $R^G = (1 - \tau)R^B$, where τ is the tax rate for positive profits.

The class of the simplest DTC utilities (12) is narrow but it contains preferences that are consistent with Allais paradox [Allais, 1979]. One can easily check that the utility (12) is consistent with common consequence effect (as defined in Allais [1979], Starmer [2000] whenever $R^G < 1/1.39$, $0.89R^B + 0.5R^G > 1$, $R^B > 40R^G$. Such R^B and R^G apparently exist. Also, utility (12) is consistent with common ratio effect if $p^* \in (1/4, 3/4)$ and $R^B > 1.25$, e.g. $R^B = 3/2$, $R^G = 1/2$, $p^* = 1/2$.

3.2. The dynamic extension

The model (c) in the previous Subsection may not seem robust to extending the horizon beyond the period $t = 1$. It turns out, however, that one can set the model in the infinite horizon framework and still obtain the Yaari utility (12). Suppose that the agent receives uncertain payoffs X_t at each period t. The payoffs X_t are independent identically distributed random variables with distribution function $F_X(\cdot)$ known to the agent. The expected value of X is finite: $|\int_{-\infty}^{+\infty} x dF_X(x)| < \infty$.

At each period t, the agent can hold non-interest bearing cash M_t and invest in risk-free bonds S_t with gross interest rate R_s. The agent can also borrow L_t at a gross interest rate R_l. The bonds have one-period maturity and the loans must be repaid next period, too. In addition to (7), we assume that the return on bonds is greater than one on money

$$R_s > 1. \tag{13}$$

The agent maximizes expected discounted cash flows C_t:

$$v = E \sum_{t=0}^{\infty} \Delta^{-t} C_t, \tag{14}$$

by choosing $M_{t+1} \geq 0$, $L_{t+1} \geq 0$, $S_{t+1} \geq 0$ and C_t under the following constraint

$$M_{t+1} = M_t - R_l L_t + R_s S_t + L_{t+1} - S_{t+1} + X_t - C_t \tag{15}$$

initial conditions

$$M_0 = M \geq 0,\ L_0 = L \geq 0,\ S_0 = S \geq 0,$$

and the no-Ponzi-game condition

$$\text{Prob}\left\{\overline{\lim_{t\to\infty}} \Delta^{-t} L_t = 0\right\} = 1. \tag{16}$$

The agent chooses whithdrawals C_t before realization of X_t and therefore can only rely upon information on state variables M_t, L_t, S_t. In the meanwhile, the choice of M_{t+1}, L_{t+1}, S_{t+1} is made after revelation of X_t and may depend upon X_t as well as on M_t, L_t, S_t. We do not impose non-negativity constraint on C_t. If the agent faces low M_t, S_t low expected value of X_t or high indebtedness L_t, then she has to plan losses (or try to attract additional capital e.g. via issue of new equity or sales of assets).

Proposition 4: *If* (7), (13) *hold, the stochastic programming problem* (14)–(16) *has a solution. The value of the objective function* (14) *as a function of the initial conditions satisfies the Bellman equation*

$$v(M, L, S) = \max_{C} \left\{ C + \Delta^{-1} E_X \max_{M', L', S' \geq 0} v(M', L', S') \right\} \tag{17}$$

where the inside maximum is taken subject to $M' = M - R_l L + R_s S + L' - S' + X - C$ and the random variable X has a c.d.f. $F_X(\cdot)$.

Proposition 5: *For both stochastic programming problem* (14)–(16) *and dynamic programming problem* (17) *there exists the same unique solution which is as follows. The Bellman function is*:

$$v(M, L, S) = M - R_l L + R_s S + U(X)\Delta/(\Delta - 1)$$

where $U(X)$ is given by (9). *The control variables are*

$$M' = 0, L' = [\bar{C} - X]_+, S' = [X - \bar{C}]_+, C = \bar{C} + M - R_l L + R_s S, \tag{18}$$

where $\bar{C}$ is determined from condition

$$F_X(\bar{C} - 0) \leq (\Delta - R_s)/(R_l - R_s) \leq F_X(\bar{C}). \tag{19}$$

If $F_X(\cdot)$ is continuous, (19) *is becomes*

$$F_X(\bar{C}) = (\Delta - R_s)/(R_l - R_s).$$

Thus, the Bellman function in a model with infinite horizon is the sum of combination of state variables $M - R_l L + R_s S$ and the present value of getting DTC utility (9) every period ad infinitum.

4. Evaluation of lotteries under non-linear financial contracts

A natural question emerges how broad is the class of DTC functionals that can be derived as an indirect utility of an agent that faces financial imperfections. Formally, we can ask whether some other DTC utilities can be derived if we introduce arbitrary schedules of interest rates as functions of amount invested/borrowed. Suppose that the gross interest payments are

$I(\xi)\Delta$ where ξ is the net savings (negative if borrowings), Δ is the gross discount rate and $I(\xi)$ is the interest rate schedule normalized by discount factor. In the previous Subsection we considered the simplest case of bid-ask spread $I(\xi) = \bar{I}_0(\xi; R^B, R^G)$ where

$$\bar{I}_0(\xi; R^B, R^G) = \begin{cases} R^G\xi & \text{if } \xi \geq 0. \\ R^B\xi & \text{if } \xi < 0 \end{cases} \tag{20}$$

Firms (as well as households) often face interest rates that not only differ for positive and negative net financial position but also depend on amounts saved or borrowed. Let us extend our analysis to the case of arbitrary $I(\xi)$. The expected discounted earnings are $U = C + \int_{-\infty}^{+\infty} I(x - C)dF_X(x)$. Again, the timeline is crucial. If the agent chooses C knowing X (similar to (a) in figure 1) then the agent remains risk-neutral $U = EX + \max_\xi\{I(\xi) - \xi\}$. If C cannot be varied at all (case (b) in figure 1), then agent evaluates the projects according to an expected utility functional. The agent maximizes expectation of non-linear utility function

$$u(x) = I(x - C). \tag{21}$$

Apparently, such an agent is risk-averse whenever $I(\xi)$ is concave.

If agent chooses C before observing X (case (c)) then the expected discounted earnings are

$$U(X) = \max_C \left\{ C + \int_0^1 I(\Phi_X(p) - C)\, dp \right\}, \tag{22}$$

where $\Phi_X(\cdot)$ is the IDF of X. We already know that for a particular case (20) with $R^G < 1 < R^B$ the utility (22) is a Yaari one with the weight function (12). The problem that we will address now is whether there exist other interest rate schedules that generate Yaari's utility. The answer to this question is negative. It turns out that no other DTC functional can be derived as an indirect utility in the maximization problem (22).

Theorem 3: *Let $I(\xi)$ be continuous and differentiable on $(-\infty, +\infty)$ except maybe a countable set. Let us also assume that there exist (finite or infinite) limits $R_+ = \lim_{\xi\to\infty} I(\xi)/\xi$ and $R_- = \lim_{\xi\to-\infty} I(\xi)/\xi$. Then utility (22) is a DTC one (1) if and only if $R_+ \leq 1 \leq R_-$ and there exists a real number b such that $I(\xi) = \bar{I}_b(\xi; R_-, R_+)$ where*

$$\bar{I}_b(\xi; R_-, R_+) = \begin{cases} b + R_+(\xi - b) & \text{if } \xi \geq b \\ b + R_-(\xi - b) & \text{if } \xi < b. \end{cases} \tag{23}$$

The Theorem complements the results obtained in the Subsection 3.1, where we showed that all risk-averse DTC utilities (1) with a weight function that takes only two values R^B and R^G can be derived as preferences of a risk-neutral agent that faces a bid-ask spread in the interest rates. Theorem 3 essentially says that no other DTC utility can be derived in such a model. No interest rate schedule that satisfies the technical conditions of the

Theorem can result in a DTC utility except the two-rate interest schedule (23) which is essentially equivalent to the bid-ask spread (20) we have already studied. Althouth (23) seems to be more general than (20) (which is only a partial case at $b = 0$), the DTC utility obtained by substituting (23) into (22) does not depend on b. Indeed, $\bar{I}_b(\xi; R_-, R_+) = b + \bar{I}_0(\xi - b; R_-, R_+)$. Hence, substituing $\tilde{C} = C + b$, we obtain

$$U(X) = \max_C C + b + \int_0^1 I(\Phi_X(p) - C - b)dp = \max_{\tilde{C}} \tilde{C} + \int_0^1 I(\Phi_X(p) - \tilde{C})\, dp$$

which is the same DTC utility as in the Subsection 3.1 with $R^G = R_+$ and $R^B = R_-$.

It is also worth emphasizing that only risk-averse utilities have microfoundations. The DTC utilities with the weight function (12) and $R^G > R^B$ cannot be derived in the form (9). Whenever $R_+ > R_-$, there is no finite solution to the maximization problem in (22).[8] This is again very different from the expected utility case: (21) implies that for every expected utility functional there exist an interest rate schedule. In particular, every risk loving expected utility can be generated by a convex $I(\cdot)$ that can be found from (21).

5. Conclusions

The main contribution of the paper is to show that a certain class of dual utilities can be obtained as an indirect utility of a risk-neutral agent that faces a bid-ask spread in the credit market. These utilities are the simplest risk-averse rank-dependent utilities which are parameterized by two numbers: the cutoff point that separates the ‘good’ and the ‘bad’ states of nature and the relative penalty for being in a bad state. The model is simple: the agent has to take the decision on the first-period withdrawals before she learns the realization of stochastic returns. If she wants to withdraw too much today, she has to borrow which will result in high interest payments and therefore low consumption tomorrow. If she withdraws too little, she would have to save and get returns tomorrow at a relatively low deposit rate. It is the agent’s ability to adjust the first-period withdrawal knowing the distribution of payoffs but not the actual realization that makes her evaluation of the lottery a risk-averse dual utility functional. If the agent were not able to adjust, her preferences would be described by a risk-averse expected utility. Vice versa, if the agent were able to adjust the withdrawals after observing the realization, she would remain risk-neutral.

We also show that no DTC utility outside this particular class has such microfoundations even if we allow for arbitrary non-linear financial contracts. Only simplest risk-averse rank-dependent utilities can be obtained as a solution to an optimization problem. This result reveals a striking difference between expected utility and dual theory of choice. Any expected utility can be derived as preferences of an agent under financial imperfections in the model where agent cannot vary the first-period consumption. In the world where agent adjusts her first-period consumption, things are very different. Under arbitrary non-linear interest schedules, the agent still has a constant absolute risk aversion, but not necessarily a constant relative risk aversion. It turns out that the only interest rate schedule that results in a DTC utility is the bid-ask spread studied above where the agent pays constant interest rate on loans and gets a constant though lower rate on deposits.

Another contribution of the paper is generalization of the dual theory of choice for unbounded lotteries with finite means. The original Yaari's formulation cannot be extended to the unbounded case. The representation form that we get is rather similar to that of Roell [1987] though the latter was also obtained for uniformly bounded lotteries.

We believe that our results justify the growing popularity of DTC as an appropriate (and very simple) tool for analysis of firms' decision-making under risk. Being tested on individuals, DTC has performed rather poorly. However, it has not yet been empirically tested on firms or banks. There is some evidence that firms' and banks' are risk-averse. Rose [1989], Davidson et al. [1992], Park and Antonovitz [1992] and many other authors prove that firms indeed seek insurance and diversification. On the other hand, only risk-averse *expected* utility has been tested on firms; it remains unknown whether DTC would perform better or worse in such tests.

Appendix

Proofs

Proof of Proposition 1. Let us first prove that the certainty equivalent exists. Take an arbitrary random variable X. According to A6, there exist such real a, A that $C(a) \prec X \prec C(A)$. Continuity (A3) and monotonicity (A4) imply that there exists such real $\xi \in (a, A)$ that $C(\xi) \sim X$. Hence, for each X we have found the certainty equivalent $\xi = U(X)$. By definition, if a random variable takes the same value in all states of nature, its certainty equivalent equals this value: $U(C(\xi)) = \xi$.

Proving $U(\alpha X) = \alpha U(X)$ is trivial. Since $X \sim C(U(X))$, A5 implies $\alpha X \sim C(\alpha U(X))$ for every $\alpha \in [0, 1]$ (take $Y = C(U(X))$ and $Z = C(0)$). Hence $U(\alpha X) = \alpha U(X)$ for every $\alpha \in [0, 1]$. Now take arbitrary X and $\alpha > 1$. Since $U(\frac{1}{\alpha}\alpha X) = \frac{1}{\alpha}U(\alpha X)$ (indeed, $\frac{1}{\alpha} < 1$), we obtain $U(\alpha X) = \alpha U(X)$.

The last step is to prove that certainty equivalence is additive. Let us first prove that for any real b, $U(X + b) = U(X) + b$. A5 implies that since $X \sim C(U(X))$, then $\frac{X+b}{2} \sim \frac{C(U(X))+b}{2}$ (take $Y = C(U(X)$, $Z = C(b)$, $\alpha = 1/2$). Hence $U(\frac{X+b}{2}) = U(\frac{C(U(X))+b}{2}) = \frac{C(U(X))+b}{2}$. Using $U(\alpha X) = \alpha U(X)$ we obtain $U(X + b) = U(X) + b$. Now consider arbitrary comonotonic X and Y. Again, $X \sim C(U(X))$ implies $\frac{X+Y}{2} \sim \frac{C(U(X))+Y}{2}$. Hence $U(X + Y) = U(C(U(X)) + Y)$. We just proved $U(C(U(X)) + Y) = U(X) + U(Y)$, therefore $U(X + Y) = U(X) + U(Y)$. □

Proof of Lemma 1. First if $Y = \alpha X$ then $\Phi_Y = \alpha\Phi_X$ and if X and Y are comonotonic then $\Phi_{X+Y} = \Phi_X + \Phi_Y$. So whenever certainty equivalence functional defined on IDFs is linear the preference relation satisfies A1–A6.

To prove the 'only if' part we need to use neutrality axiom A1. Consider arbitrary $\Phi^1, \Phi^2 \in M$, $\Phi^2 = \alpha\Phi^1$. Take any random variable X such that $\Phi_X = \Phi^1$ and then consider random variable αX. By definition $\Phi^2 = \Phi_{\alpha X}$. Then for any random variable Y such that $\Phi_Y = \Phi^2 = \Phi_{\alpha X}$ neutrality requires $U(Y) = U(\alpha X) = \alpha U(X)$.

Similarly, consider $\Phi^1, \Phi^2, \Phi^3 \in M : \Phi^3 = \Phi^1 + \Phi^2$. Then take any $Z : \Phi_Z = \Phi^3$ and define X and Y in the following way: for every real $z \in [\Phi_3(0), \Phi_3(1)]$ find $\hat{p}(z)$ such that $\Phi^3(\hat{p}) = z$. Then X takes value $\Phi^1(\hat{p}(z))$ and Y takes value $\Phi^2(\hat{p}(z))$ whenever Z takes value z. Thus we have obtained comonotonic X, Y such that $\Phi_X = \Phi^1$, $\Phi_Y = \Phi^2$ and $X + Y = Z$. For them $U(X+Y) = U(Z) = U(X) + U(Y)$. Due to A1, this will also be true for all X, Y, Z such that $\Phi_X = \Phi^1$, $\Phi_Y = \Phi^2$, $\Phi_Z = \Phi^3$. □

Proof of Theorem 1. The utility functional V is defined and linear on M. Let us extend it onto the whole space $L^1[0, 1]$. For any function $g \in L^1(0, 1)$ there exists a representation $g = \lim_{n\to\infty} g_n$, where $g_n = \Phi_n^1 - \Phi_n^2$, and $\Phi_n^1, \Phi_n^2 \in M$. Indeed, the set of continuous functions is dense in $L^1(0, 1)$; the set of polynomials is dense in the set of continuous functions, and every polynomial can be represented as a difference of two monotonic functions.

Define $V(g) = \lim_{n\to\infty} V(\Phi_n^1) - V(\Phi_n^2)$. Due to linearity of V over M there is no ambiguity: $V(g)$ does not depend upon representation of g. Let $g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} \tilde{g}_n$, where $g_n = \Phi_n^1 - \Phi_n^2$, and $\tilde{g}_n = \tilde{\Phi}_n^1 - \tilde{\Phi}_n^2$. Then $\|g_n - \tilde{g}_n\| = \|\Phi_n^1 - \Phi_n^2 - \tilde{\Phi}_n^1 + \tilde{\Phi}_n^2\| \to 0$. Grouping the terms and using continuity of V over M we obtain $V(\Phi_n^1 + \tilde{\Phi}_n^2) - V(\tilde{\Phi}_n^1 + \Phi_n^2) \to 0$. Then applying linearity of V on M and re-grouping terms back we get $(V(\Phi_n^1) - V(\Phi_n^2)) - (V(\tilde{\Phi}_n^1) - V(\tilde{\Phi}_n^2)) \to 0$. Thus, every utility built upon a preference relation that satisfies A1–A6 generates a linear continuous functional on $L^1[0, 1]$. It is well known [see Yosida, 1978] that the set of all such functionals is isomorphic to conjugate space $L^\infty[0, 1]$ i.e. each utility functional can be represented as $\int_0^1 h(p)\Phi(p)dp$ where $h \in L^\infty[0, 1]$.

Monotonicity implies $h(p) \geq 0$ and certainty equivalence axiom requires normalization $\int_0^1 h(p)dp = 1$. □

Proof of Theorem 2. The proof basically follows Yaari [1987]. First let us consider two random variables with IDF $\Phi^1(p)$ and $\Phi^2(p)$ such that $\int_0^1 \Phi^1(p)dp = \int_0^1 \Phi^2(p)dp$ (equal means) and for any a holds $\|\Phi^1(p) - a\|\Phi^2(p) - a\|$. Consider $\Phi(p) = \Phi^1(p) - \Phi^2(p)$. Let us now prove that $K(q) = \int_0^q \Phi(p)dp \geq 0$ for any $q \in [0, 1]$.

Then let us divide [0, 1] into intervals (p_k, p_{k+1}) on which $\Phi(p)$ has same sign ($\Phi(p)$ changes sign at least once). Then for every sign change point p_k there exist a_k such that $\Phi^{1,2}(p) \geq a_k$ whenever $p > p_k$ and $\Phi^{1,2}(p) \leq a_k$ whenever $p < p_k$. Then using the risk aversion condition we obtain $0 \geq \|\Phi^1(p) - a_k\| - \|\Phi^2(p) - a_k\| = 2(\int_0^{p_k}((\Phi^1(p) - a_k) - (\Phi^2(p) - a_k))dp)$, i.e. $K(p_k) \geq 0$. Similarly, $K(p_{k+1}) \geq 0$. Furthermore, for an arbitrary $p \in (p_k, p_{k+1})$ we have either $K(p_k) \leq K(p) \leq K(p_{k+1})$ (if $\Phi(p)$ is non-negative on (p_k, p_{k+1})) or $K(p_{k+1} \leq K(p) \leq K(p_k)$ (if $\Phi(p)$ is non-positive on (p_k, p_{k+1})).

Now when we have proved that $K(q) \geq 0$ for all $q \in [0, 1]$, we may integrate by parts: $V(\Phi^1) - V(\Phi^2) = \int_0^1 h(p)\Phi(p)dp = \int_0^1 h(p)dK(p) = h(p)K(p)|_0^1 - \int_0^1 K(p)dh(p) \geq 0$.

The non-integral term is zero since $K(0) = K(1) = 0$. The integral term is non-positive (since $h(p)$ is monotonic, $dh(p)$ is a non-positive measure and since $K(p)$ is an absolutely continuous function, the integral exists and is non-positive).

Let us now assume that there is a set $P \subset [0, 1]$ of non-zero measure on which a risk averse $h(p)$ is strictly increasing. Let us divide the set into two subsets of equal measure: $P = P_1 \oplus P_2$, $\int_{P_1} dp = \int_{P_2} dp$, such that $p_1 \leq p_2$ for all $p_1 \in P_1$, $p_2 \in P_2$. Consider two random variables ξ and η : $\Phi_\xi(p) = \Phi_\eta(p)$ for all $p \notin P$, $\Phi_\xi(p) = 1/2$ for $p \in P$ and

$\Phi_\eta(p) = 0$ if $p \in P_1$ and $H_\eta(p) = 1$ if $p_2 \in P_2$. Then $\|\Phi_\xi(p) - a\| \leq \|\Phi_\eta(p) - a\|$ for any real a. Risk aversion implies $U(\xi) \geq U(\eta)$. Hence $\frac{1}{2}\int_P h(p)dp \geq \int_{P_2} h(p)dp$, or $\int_{P_1} h(p)dp \geq \int_{P_2} h(p)dp$. This means that $h(p)$ can not be strictly increasing on P. □

Proof of Proposition 3. For a given distribution, let us calculate the value of (9) substituting $C = \Phi_X(p^*)$:

$$\begin{aligned} U(X) &= \Phi_X(p^*) - R^B \int_0^{P^*} (\Phi_X(p^*) - \Phi_X(p))\,dp + R^G \int_{p^*}^1 (\Phi_X(p) - \Phi_X(p^*))\,dp \\ &= \int_0^{p^*} R^B \Phi_X(p)\,dp + \int_{p^*}^1 R^G \Phi_X(p)\,dp = \int_0^1 h(p)\Phi_X(p)dp. \end{aligned}$$

where the weight function $h(\cdot)$ is described by (12). □

Proof of Proposition 4. The partial sum of (14) is as follows.

$$\begin{aligned} \sum_{t=\tau}^{T} \Delta^{\tau-t} C_t \;=\; & \{M_\tau - R_l L_\tau + R_s S_\tau\} + \sum_{t=\tau}^{T} \Delta^{\tau-t} X_t \\ & - (\Delta - 1) \sum_{t=\tau+1}^{T} \Delta^{\tau-t} M_t - (R_l - \Delta) \sum_{t=\tau+1}^{T} \Delta^{\tau-t} L_t \\ & - (\Delta - R_s) \sum_{t=\tau+1}^{T} \Delta^{\tau-t} S_t + \Delta^{\tau-T} L_{T+1} \\ & - \Delta^{\tau-T}(M_{T+1} + S_{T+1}) \end{aligned}$$

Under assumptions about interest rates (7) and no-Ponzi-game condition (16) expected value of $\sum_{t=\tau}^{T} \Delta^{\tau-t} C_{t+1}$ is bounded from above by $M_t - R_l L_t + R_s S_t + const$. Therefore the problem has a solution and the Bellman function is well-defined. Let us define a state $Z_t = \{M_t, L_t, S_t\}$. Then the agent's strategy is a quadruple of Borel functions $\Omega = \{\tilde{M}(t, Z, X), \tilde{L}(t, Z, X), \tilde{S}(t, Z, X), \tilde{C}(t, Z)\}$, where $\tilde{M}(t, Z, X), \tilde{L}(t, Z, X), \tilde{S}(t, Z, X)$ are non-negative and for all M, L, S and X holds $\tilde{M}(t, \{M, L, S\}, X) = M + X - R_l L + R_s S + \tilde{L}(t, \{M, L, S\}, X) - \tilde{S}(t, \{M, L, S\}, X) - \tilde{C}(t, \{M, L, S\})$. The strategy defines a Markov process $Z_{t+1} = \{\tilde{M}(t, Z_t, X_t), \tilde{L}(t, Z_t, X_t), \tilde{S}(t, Z_t, X_t)\} = G_\Omega(t, Z_t, X_t)$ and a series of random variables $C_t = \tilde{C}(t, Z_t)$. Therefore for each strategy Ω and initial conditions Z one can calculate the expected net present value $v_\Omega(\tau, Z) = E\{\sum_{t=\tau}^{T} \Delta^{\tau-t} C_t | Z_\tau = Z\}$. We have just shown that v_Ω is bounded and the maximization problem $\max_\Omega J_\Omega(\tau, Z)$ is well-defined. The Kolmogorov equation for $v_\Omega(t, Z)$ is as follows:

$$v_\Omega(t, Z) = \tilde{C}(t, Z) + \Delta^{-1} J_\Omega(t + 1, G_\Omega(t, Z, X)).$$

By definition the optimal strategy $\hat{\Omega}$ satisfies $v_{\hat{\Omega}}(t, Z) \geq v_{\Omega}(t, Z)$ for all t, Z and Ω. Therefore it also satisfies the Bellman equation

$$J_{\Omega}(t, Z) = \sup_{\Omega(t,\cdot)} \tilde{C}(t, Z) + \Delta^{-1} v_{\Omega}(t+1, G_{\Omega}(t, Z, X)).$$

Apparently, time index t can be omitted. Substituting for Z and Ω, we obtain (17). □

Proof of Proposition 5. In order to solve the Bellman equation, we shall introduce $C = C - (M - R_l L + R_s S)$. The Bellman equation can then be re-written as

$$v(M, L, S) = M - R_l L + R_s S + \max_{\Phi} \left\{ \Phi + \Delta^{-1} E_X \max_{\substack{M', L', S' \geq 0 \\ M' = L' - S' + X - \Phi}} v(M', L', S') \right\}.$$

The Bellman function is therefore linear:

$$v(M, L, S) = M - R_l L + R_s S + (\Delta - 1)^{-1} \Delta U,$$

where

$$U(X) = \max_{\bar{C}} \left\{ \bar{C} + \Delta^{-1} E_X \max_{\substack{L', S' \geq 0 \\ L' - S' + X - \bar{C} \geq 0}} L' - S' + X - \bar{C} - R_l L' + R_s S' \right\}.$$

According to (7), the expression $L' - S' + R_l L' + R_s S'$ increases whenever L' and S' decrease by the same amount. Therefore $L' = [\bar{C} - X]_+$ and $S' = [X - \bar{C}]_+$. Then

$$U(X) = \max_{\bar{C}} \left\{ \bar{C} - R^B \int_{-\infty}^{\bar{C}} (\bar{C} - x) dF_X(x) + R^G \int_{\bar{C}}^{\infty} (x - \bar{C}) dF_X(x) \right\},$$

where $R^B = R_l/\Delta > 1$ and $R^G = R_s/\Delta < 1$. The first- and second-order conditions imply (19). □

Proof of Theorem 3. First, let us prove that the maximization problem (22) has a finite solution only if $R_+ \leq 1 \leq R_-$. Indeed, if $R_+ > 1$, the agent would choose $C = \infty$ and would get an infinite utility, while if $R_- < 1$, infinite utility is achieved by taking $C = -\infty$.

All DTC utilities (1) have both constant absolute and constant relative risk-aversion. We shall prove now that in order for (22) having a constant absolute and constant relative risk-aversion, the interest payment schedule $I(\xi)$ must be represented in the form (23).

It is clear that (22) has constant absolute risk aversion: for any real b

$$V(\Phi + b) = b = \max_{C} \left\{ C - b + \int_0^1 I(\Phi(p) - (C - b)) dp \right\} = b + V(\Phi).$$

Let us now check whether (22) has constant relative risk aversion. For any real $\alpha > 0$ we should have $\alpha^{-1}V(\alpha\Phi) = V(\Phi)$. Apparently,

$$\alpha^{-1}V(\alpha\Phi) = \max_C \left\{ C + \int_0^1 \alpha^{-1} I(\alpha(\Phi(p) - C))\, dp \right\}. \tag{24}$$

First, we shall consider $I(\xi)$ such that for any real non-negative α holds $I(\Phi\xi) = \alpha I(\xi)$. This implies

$$I(\xi) = \begin{cases} R_+\xi & \text{if } \xi \geq 0 \\ R_-\xi & \text{if } \xi < 0. \end{cases}$$

If $R_+ < R_-$ then it is the DTC utility derived in the Subsection 3.1 (it is a particular case of (23) for $a = 0$). If $R_+ = R_- = 1$ then it is the case of risk-neutrality.

Now we shall see what happens if for some $\alpha > 0$ and ξ we have $I(\alpha\xi) \neq \alpha I(\xi)$. Take this α and introduce $J(\xi) = \alpha^{-1} I(\alpha\xi)$. Then (24) takes the form

$$\alpha^{-1}V(\alpha\Phi) = \max_C \left\{ C + \int_0^1 J(\Phi(p) - C)\, dp \right\}. \tag{25}$$

Let $C_I(\Phi)$ and $C_J(\Phi)$ be solutions of the maximization problems in (22) and (25), correspondingly. Notice that since $R_+ \leq 1 \leq R_- < \infty$, $C_J(\Phi)$ is finite even at $\alpha \to \infty$.

Assuming constant relative risk aversion $\alpha^{-1}V(\alpha\Phi) = V(\Phi)$, we obtain

$$C_J(\Phi) + \int_0^1 J(\Phi(p) - C_J(\Phi))\, dp = C_I(\Phi) + \int_0^1 I(\Phi(p) - C_I(\Phi))\, dp.$$

This equality should hold for any Φ. Consider $\tilde{\Phi}$ sufficiently close to Φ. According to the envelope theorem,

$$\begin{aligned} V(\tilde{\Phi}) - V(\Phi) &= \int_0^1 I'(\Phi(p) - C_I(\Phi))\delta(p)dp + O(\|\delta\|^2) \\ \alpha^{-1}V(\alpha\tilde{\Phi}) - \alpha^{-1}V(\alpha\Phi) &= \int_0^1 J'(\Phi(p) - C_J(\Phi))\delta(p)dp + O(\|\delta\|^2) \end{aligned}$$

where $\delta = \tilde{\Phi} - \Phi$ is small. Hence, $\int_0^1 J'(\Phi(p) - C_J(\Phi))\delta(p)dp = \int_0^1 I'(\Phi(p) - C_I(\Phi))\delta(p)\, dp + O(\|\delta\|^2)$. Since the set of all possible deviations $\delta(p)$ is sufficiently rich we should have $J'(\xi) = I'(\xi + b)$ almost everywhere. Here $b = C_J(\Phi) - C_I(\Phi)$.

The condition $J'(\xi) = I'(\xi + b)$ implies $J(\xi) = J(0) + I(\xi + b) - I(b)$. Substituting into $\alpha^{-1}V(\alpha\Phi) = V(\Phi)$ we obtain $J(0) = I(b) - b$. Therefore $J(\xi) = I(\xi + b) - b$.

Thus, utility (22) has constant relative risk aversion only if for every $\alpha > 0$ there exists a real b such that $J(\xi) = \alpha^{-1}I(\alpha\xi) = I(\xi + b) - b$ for all ξ. Let us see what happens if

$\alpha \to \infty$. For a given $\eta > 0$

$$\eta \lim_{\alpha\xi \to +\infty} \frac{I(\alpha\eta)}{\alpha\eta} = \eta R_+ = I(\eta + b) - b.$$

Thus, $I(\eta + b) = b + R_+\eta$. Similarly, for $\eta < 0$ we have $I(\eta + b) = b + R_-\eta$. Substituting $\xi = \eta + b$, we obtain the formula (23).

We have proved that utility (22) can be a DTC one only if the interest payment schedule is (23). Let us check whether the condition is also (23) sufficient. Let us take arbitrary b and $R_+ \leq 1 \leq R_-$. Then making straightforward calculations one can show that the utility (22) is the one introduced in the Subsection 3.1, i.e. the utility (1) with the weight function (12) where $R^G = R_+$ and $R^B = R_-$. □

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Notes

1. Strictly speaking, L^1 is a space of *classes* of functions which may differ on a set of measure zero. According to our definition of IDF, for each class we take the representative of the class that is upper semi-continuous or, which is the same, right semi-continuous. However, the choice of a particular representative function within the class does not matter.
2. Hereinafter $F_X(\cdot)$ and $\Phi_X(\cdot)$ are cumulative distribution function and inverse distribution function of random variable X.
3. Yaari's proof is different. Its idea is to 'lay Neuman-Morgenstern result on its side' [Yaari, 1987].
4. As shown in Rotchild and Stiglitz [1970] for such X and Y there exists a noise ξ that meets conditions of Definition 3 so that Y and $X + \xi$ have the same distribution.
5. Throughout the paper we stick to the conventional choice-under-risk setting: an individual evaluates lotteries to maximize consumption. On the other hand, the model also describes a firm. For the firm's case, Δ would be internal rate of return (or cost of internal finance), while R_l would be cost of external finance, and R_s outside investment opportunities.
6. Hereinafter $\xi_+ = \max\{\xi, 0\}$.
7. Hereinafter $F(x - 0) = \lim_{x' \to x, x' < x} F(x')$.
8. The risk-loving utility functionals with $R^G > R^B$ are similar to the Pangloss value functional in Krugman [1998].

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