Contracts HA #1 solutions

Bolton and Dewatripont, Contract Theory, Exercises 1,3,5 for Chapter 2.

1. Government agency utility

$$U_G = B(q) - P;$$

Firm of type c profit:

$$\Pi_C = P - cq.$$

a). $c \in \{c_L, c_H\}$, $P(c = c_L) = \beta$

Agency problem:

$$EU_G = \beta(B(q_L) - P_L) + (1-\beta)(B(q_H) - P_H) \to \max$$

IC$_L$: $P_L - c_L q_L \geq P_H - c_L q_H;$
IC$_H$: $P_H - c_H q_H \geq P_L - c_H q_L;$
IR$_L$: $P_L - c_L q_L \geq 0;$
IR$_H$: $P_H - c_H q_H \geq 0.$

IR$_H$, **IC$_L$** are binding =>

$$P_H = c_H q_H; \quad P_L = (c_H - c_L)q_H + c_L q_L.$$

Optimal contract:

$$EU_G = \beta(B(q_L) - (c_H - c_L)q_H - c_L q_L) + (1-\beta)(B(q_H) - c_H q_H) \to \max$$

Second-best:

$B'(q_L^*) = c_L;$

$B'(q_H^*) = \int u_E(W_0 + T_H)dF_\varepsilon = \int u_E(W_0 + x_L(x_L, T_L, \varepsilon))dF_\varepsilon$

b). First-best:

$$B(q_i) - c_i q_i \to \max,\ i \in \{L, H\} \Rightarrow B'(q_i^{FB}) = c_i.$$

It is readily seen that:

$$q_L^* = q_L^{FB}, \quad q_H^* < q_H^{FB}.$$

c). $c \sim U[\underline{c}, \overline{c}]$.

Agency problem:

$$EU_G = \int_{\underline{c}}^{\overline{c}} (B(q(c)) - P(c))dF(c) \to \max_{\{q(c),P(c)\}}$$

IC: $P(c) - cq(c) = \max_{\hat{c}}(P(\hat{c}) - cq(\hat{c}))$, $\forall\, c \in [\underline{c}, \overline{c}]$;

IR: $P(c) - cq(c) \geq 0$, $\forall\, c \in [\underline{c}, \overline{c}]$.

The hazard rate for the uniform distribution is non-decreasing, so we can use Mirrlees approach. Firm of type c welfare

$$W(c) = \max_{\hat{c}}(P(\hat{c}) - cq(\hat{c})).$$

IR($\overline{c}$) is binding => $P(\overline{c}) = \overline{c}\, q(\overline{c})$ => $W(\overline{c}) = 0$.

Firm welfare change:

$$\frac{dW}{dc} = \frac{\partial W}{\partial c} = -q(c) \quad => \quad W(c) - W(\overline{c}) = \int_{\overline{c}}^{c} (-q(x))dx \quad => \quad W(c) = -\int_{\overline{c}}^{c} q(x)dx\,.$$

From **IC**: $P(c) - cq(c) = W(c) => P(c) = W(c) + cq(c) = cq(c) - \int_{\overline{c}}^{c} q(x)dx$.

Substituting into EU_G we get

$$EU_G = \int_{\underline{c}}^{\overline{c}} [B(q(c)) - (cq(c) - \int_{\overline{c}}^{c} q(x)dx)]dF(c) = \frac{1}{\overline{c} - \underline{c}} \left(\int_{\underline{c}}^{\overline{c}} \left[B(q(c)) - cq(c) \right] dc + \int_{\underline{c}}^{\overline{c}} (\underline{c} - c)q(c)dc \right) =$$

$$= \frac{1}{\overline{c} - \underline{c}} \int_{\underline{c}}^{\overline{c}} \left[B(q(c)) - (2c - \underline{c})q(c) \right] dc \to \max_{q(c)}$$

Maximizing the term under the integral w.r.t. $q(c)$, we get:

$$B'(q(c)) = 2c - \underline{c}.$$

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3. Monopolist's profit

$$\Pi = P - s^2;$$

Consumer of type θ's utility:

$$U_\theta = \begin{cases} \theta s - P, & if\ buys \\ 0, & else \end{cases}.$$

a). First-best:

$$\theta s - s^2 \to \max => s^{FB}(\theta) = \frac{\theta}{2}.$$

b). $\theta \in \{\theta_L, \theta_H\}$, $P(\theta = \theta_L) = \beta$

Firm's problem:

$$E\Pi = \beta(P_L - s_L^2) + (1 - \beta)(P_H - s_H^2) \to \max$$

IC$_L$: $\theta_L s_L - P_L \geq \theta_L s_H - P_H$;
IC$_H$: $\theta_H s_H - P_H \geq \theta_H s_L - P_L$;
IR$_L$: $\theta_L s_L - P_L \geq 0$;
IR$_H$: $\theta_H s_H - P_H \geq 0$.

IR$_L$, **IC$_H$** are binding =>

$$P_L = \theta_L s_L; \qquad P_H = \theta_H(s_H - s_L) + \theta_L s_L.$$

Optimal contract:

$$EU_G = \beta(\theta_L s_L - s_L^2) + (1 - \beta)(\theta_H(s_H - s_L) + \theta_L s_L - s_H^2) \to \max$$

Second-best: $s_H^* = \frac{\theta_H}{2}$; $\quad s_L^* = \frac{1}{2}\left(\theta_L + \frac{1-\beta}{\beta}(\theta_L - \theta_H) \right)$

$s_H^* = s_H^{FB}$, $\quad s_L^* < s_L^{FB}$.

Informational rent: $R(\theta) = U_\theta(s_\theta^*) - U_\theta(s_\theta^{FB}) = U_\theta(s_\theta^*)$.

$R(\theta_L) = 0$, $R(\theta_H) = \frac{1}{2}(\theta_H - \theta_L)(\frac{1}{\beta}\theta_L - \frac{1-\beta}{\beta}\theta_H)$ (if both types are served)

c). $\theta \sim U\,[0, 1]$.

Monopoly problem:

$$\Pi = \int_0^1 (P(\theta) - s(\theta)^2) dF(\theta) \to \max_{\{s(\theta), P(\theta)\}}$$

IC: $\theta s(\theta) - P(\theta) = \max_{\hat\theta}(\theta s(\hat\theta) - P(\hat\theta))$, $\forall\ \theta \in [0, 1]$;

IR: $\theta s(\theta) - P(\theta) \geq 0$, $\forall\ \theta \in [0, 1]$.

Consumer of type θ 's welfare

$$W(\theta) = \max_{\hat\theta}(\theta s(\hat\theta) - P(\hat\theta)).$$

IR(0) is binding => $W(0) = 0$ => $P(0) = 0$, $s(0) = 0$.

Consumer's welfare change:

$$\frac{dW}{d\theta} = \frac{\partial W}{\partial \theta} = s(\theta) \quad \Rightarrow \quad W(\theta) = \int_0^\theta s(x)dx + W(0) = \int_0^\theta s(x)dx .$$

From **IC**: $\theta s(\theta) - P(\theta) = W(\theta) \Rightarrow P(\theta) = \theta s(\theta) - W(\theta) = \theta s(\theta) - \int_0^\theta s(x)dx$.

Substituting into Π we get

$$\Pi = \int_0^1 \left(P(\theta) - s^2(\theta)\right) d\theta = \int_{\underline{c}}^{\bar{c}} \left(\theta s(\theta) - \int_0^\theta s(x)dx - s^2(\theta)\right) d\theta =$$

$$= \int_0^1 \left((2\theta - 1)s(\theta) - s^2(\theta)\right) d\theta \to \max_{s(\theta)}$$

Maximizing the term under the integral w.r.t. $s(\theta)$, we get:

$$s^*(\theta) = \frac{2\theta - 1}{2} = \theta - \frac{1}{2}.$$

Substituting in the equation for P

$$P(\theta) = \theta s(\theta) - \int_0^\theta s(x)dx = \frac{\theta^2}{2} = \frac{1}{2}\left(s + \frac{1}{2}\right)^2.$$

So, the optimal contract schedule is $P(\theta) = \frac{1}{2}\left(s + \frac{1}{2}\right)^2$.

5.a. Entrepreneur's expected utility:

$$Eu_i = \begin{cases} \int_{-\infty}^{+\infty} u_E(W_0 + x(\theta_i+\varepsilon)+T)dF_\varepsilon, & \textit{if accepts offering} \\ \int_{-\infty}^{+\infty} u_E(W_0+\theta_i+\varepsilon)dF_\varepsilon, & \textit{if rejects} \end{cases}, \quad i \in \{L, H\}.$$

b. First-best:

Denote $P_i(x, T, \varepsilon) = W_0 + x(\theta_i + \varepsilon) + T$, then we can find FB solving the problem:

$$U_I \to \max_{\{x,T\}},$$

$$0 \le x_i \le 1,$$

$$\mathbf{IR}_i: \quad \int u_E(P_i(x_i, T_i, \varepsilon))dF_\varepsilon \ge \int u_E(W_0+\theta_i+\varepsilon)dF_\varepsilon, \; i \in \{L, H\}.$$

Lagrange function for this problem is

$$L = EU_I + \lambda\,\mathbf{IR}_i + \gamma x = (1 - x_i)\theta_i - T_i + \lambda\int[u_E(P_i(x_i,T_i,\varepsilon)) - u_E(W_0+\theta_i+\varepsilon)] + \gamma x_i,$$

(we omit the term corresponding to the constraint $x \le 1$ which is not binding in optimum).

$$\frac{\partial W}{\partial x} = -\theta_i + \lambda\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))(\theta_i+\varepsilon)dF_\varepsilon + \gamma = 0 \qquad (1)$$

$$\frac{\partial W}{\partial T} = -1 + \lambda\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))dF_\varepsilon = 0 \qquad (2)$$

From (1) => $\theta_i\left(\lambda\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))dF_\varepsilon - 1\right) + \lambda\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))\varepsilon dF_\varepsilon + \gamma = 0$, and from (2)

$\lambda\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))dF_\varepsilon = 1$ => $\lambda\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))\varepsilon dF_\varepsilon = -\gamma$

Since $E\varepsilon = 0$ and u_E' is decreasing then $\int_{-\infty}^{+\infty} u'_E(P_i(x,T,\varepsilon))\varepsilon dF_\varepsilon < 0$ => $\gamma > 0$ => the constraint $x = 0$ is binding => the bank fully insures the entrepreneur; the payment T is determined from the entrepreneur's IR constraint:

$$u_E(W_0 + T_i) = \int_{-\infty}^{+\infty} u_E(W_0+\theta_i+\varepsilon)dF_\varepsilon$$

Notice that the capital structure is 100% outside equity and no debt since there is no moral hazard problem and the issue is allocation of risk between risk-neutral investor and risk-averse entrepreneur.

c. Screening problem:

$$EU_I = \beta((1 - x_L)\theta_L - T_L) + (1 - \beta)((1 - x_H)\theta_H - T_H) \to \max_{\{x_L, x_H, T_L, T_H\}},$$

$$0 \le x_L, x_H \le 1,$$

$\mathbf{IC_L}$: $\int u_E(P_L(x_L,T_L,\varepsilon))dF_\varepsilon \ge \int u_E(P_L(x_H,T_H,\varepsilon))dF_\varepsilon$

$\mathbf{IC_H}$: $\int u_E(P_H(x_H,T_H,\varepsilon))dF_\varepsilon \ge \int u_E(P_H(x_L,T_L,\varepsilon))dF_\varepsilon$

IR$_L$: $\int u_E(P_L(x_L, T_L, \varepsilon))dF_\varepsilon \geq \int u_E(W_0 + \theta_L + \varepsilon)dF_\varepsilon$

IR$_H$: $\int u_E(P_H(x_H, T_H, \varepsilon))dF_\varepsilon \geq \int u_E(W_0 + \theta_H + \varepsilon)dF_\varepsilon$

Notice that the outside option (the right hand side) in IR$_i$ depends on type θ_i.

d. First note that for each type of entrepreneur one of the constraints **IC** and **IR** will be binding in optimum. Indeed, if we maximize EU_I with respect to one contract terms left hand sides of both constraints of the corresponding entrepreneur's type decline, while right hand sides remain the same. So maximum is achieved when one of the constraints become binding. The problem differs from the standard one in the Chapter two (outside options depend on type) hence we need to go through all possible cases.

If both **IR$_L$** and **IR$_H$** are binding the maximization of EU_I implies that $x_L = x_H = 0$ (see the discussion of first best above). However, in this case, $T_H > T_L$ and **IC$_L$** is violated. This allows us to make an educated guess that probably in equilibrium the binding constraints are **IC$_L$** and **IR$_H$**; but just to make sure let's check the other cases.

If **IR$_L$** and **IC$_H$** are binding then in optimum $x_H = 0$ (it can be inferred from FOC's). The utility of H-type entrepreneur is

$$\int u_E(W_0 + T_H)dF_\varepsilon$$

The L-type entrepreneur can get the same utility if he pretend to be H-type, since it doesn't depend on the type. This value is more than $\int u_E(W_0 + \theta_L + \varepsilon)dF_\varepsilon$ that he can get if reports the true type => **IC$_L$** is not satisfied in this case.

e, f. Guess that L-type entrepreneur will get first-best allocation => $x_L = 0$, $T_L = T_L^{FB}$. Then Lagrange function is:

$$L = EU_I + \lambda\, \mathbf{IR_H} + \mu\, \mathbf{IC_L} = \beta((1 - x_L)\theta_L - T_L) + (1 - \beta)((1 - x_H)\theta_H - T_H) + \gamma x_L +$$

$$+\ \lambda \int [u_E(P_H(x_H, T_H, \varepsilon)) - u_E(W_0 + \theta_H + \varepsilon)] \ +\ \mu \int [u_E(P_L(x_L, T_L, \varepsilon)) - u_E(P_L(x_H, T_H, \varepsilon))]dF_\varepsilon,$$

FOC:

$$\frac{\partial L}{\partial x_L} = -\beta\theta_L + \mu \int_{-\infty}^{+\infty} u_E'(P_L(x_L, T_L, \varepsilon))(\theta_L + \varepsilon)dF_\varepsilon + \gamma = 0$$

$$\frac{\partial L}{\partial T_L} = -\beta + \mu \int_{-\infty}^{+\infty} u_E'(P_L(x_L, T_L, \varepsilon))dF_\varepsilon = 0$$

$$\frac{\partial L}{\partial x_H} = -(1-\beta)\theta_H + \lambda \int_{-\infty}^{+\infty} u_E'(P_H(x_H, T_H, \varepsilon))(\theta_H + \varepsilon)dF - \mu \int_{-\infty}^{+\infty} u_E'(P_L(x_H, T_H, \varepsilon))(\theta_L + \varepsilon)dF_\varepsilon = 0 \quad (3)$$

$$\frac{\partial L}{\partial T_H} = -(1-\beta) + \lambda \int_{-\infty}^{+\infty} u_E'(P_H(x_H, T_H, \varepsilon))dF_\varepsilon - \mu \int_{-\infty}^{+\infty} u_E'(P_L(x_H, T_H, \varepsilon))dF_\varepsilon = 0 \quad (4)$$

The constraint $x_H \geq 0$ is not binding since either **IC$_L$** or **IC$_H$** is violated in this case (if $T_H \neq T_L$). $T_H = T_L$ means that the bank offers the same contract regardless the type of the entrepreneur, that is not optimal. Hence the H-type entrepreneur is not fully insured. Her contract (x_H, T_H) can be found from **IR$_H$**, **IC$_L$** and (3), (4).

g. The difference from the standard screening models comes from the fact that each type's outside option depends on its type, moreover the slope of the relationship between outside option and type is so steep that in equilibrium it is the low type whose IC is binding hence it is the high type whose allocation is distorted (he gets imperfect insurance).

h. In a competitive equilibrium

$$EU_I = \beta((1 - x_L)\theta_L - T_L) + (1 - \beta)((1 - x_H)\theta_H - T_H) = 0.$$

The difference is that in the first best the IR do not have to bind, and the first best can be implemented. Indeed, consider full insurance for both types $x_L = x_H = 0$ and the T_L and T_H solving the IC constraints. As $P_i=W_0+T_i$, both ICs can be satisfied if and only if $T_L=T_H=T$. Substituting $x_i = 0$ and $T_i=T$ into the condition $EU_I = 0$, we find $T=\beta\theta_L + (1 - \beta)\theta_H$. Then substituting back into IR_i we find that both agents earn positive rents.

Problem 6.

We assume (as in 3.1) that firm has all bargaining power => investor's profit is 0.

a). Let R be the only choice variable.
Then investor's expected profit is

$$U_I = \beta\theta_L R_L + (1-\beta)\theta_H R_H - I.$$

Firm's of type i expected profit is

$$W_i = \theta_i(C - R_i).$$

One can see that firms' profits are linear in R. Then the signaling with R fails since in the case when $R_H \neq R_L$ it is profitable for either H-type firm or L-type firm to pretend to be the type with lower R.

So, the pooling equilibrium will be here, when firms pay $R_H = R_L = \dfrac{I}{\widetilde{\theta}}$, where $\widetilde{\theta} = \beta\theta_L + (1-\beta)\theta_H$.

b). Now firms choose a pair (R, K), where K is the size of collateral.
The investor's expected profit is

$$U_I = \beta(\theta_L R_L + (1-\theta_L)xK_L) + (1-\beta)(\theta_H R_H + (1-\theta_H)xK_H) - I.$$

Type i firm's expected profit is

$$W_i = \theta_i(C - R_i) - (1-\theta_i)K_i.$$

In the (K, R) space, type i's indifference curves have a slope $(1-\theta_i)/\theta_i$, i.e. higher type's indifference curves are flatter.

1). Pooling equilibrium.

Both types choose the same (R, K) and the investor faces IR constraint:

$$\beta(\theta_L R + (1-\theta_L)xK) + (1-\beta)(\theta_H R + (1-\theta_H)xK) - I = 0.$$

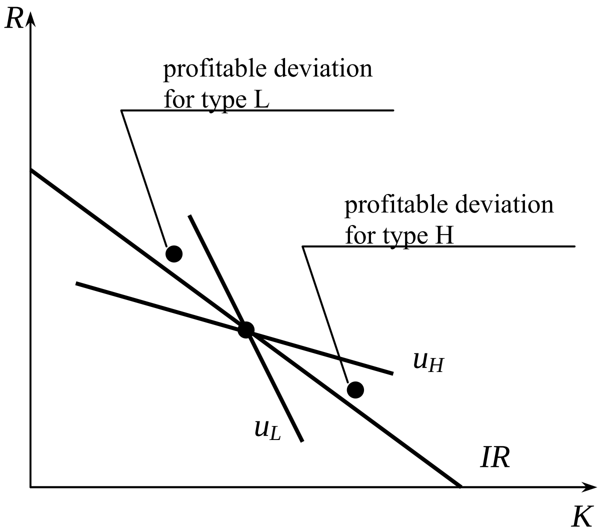
$$R = \frac{I}{\widetilde{\theta}} - \frac{1-\widetilde{\theta}}{\widetilde{\theta}}xK$$

Since $\theta_L < \widetilde{\theta} < \theta_H$ and $\dfrac{1-\theta}{\theta}$ is decreasing function then 2 cases are possible:

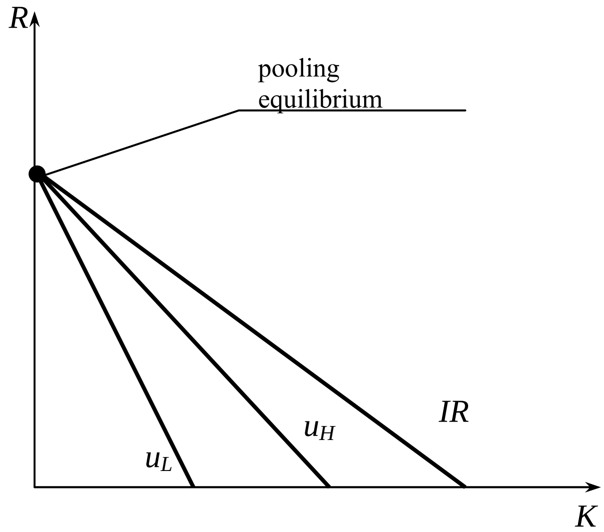
i). $\dfrac{1-\theta_H}{\theta_H} < \dfrac{1-\widetilde{\theta}}{\widetilde{\theta}}x < \dfrac{1-\theta_L}{\theta_L}$

ii). $\dfrac{1-\widetilde{\theta}}{\widetilde{\theta}}x < \dfrac{1-\theta_H}{\theta_H} < \dfrac{1-\theta_L}{\theta_L}$

These cases are depicted below.



i).



ii).

In the first case there is no pooling equilibrium since it is profitable for either type of the agent to deviate and offer better contract.

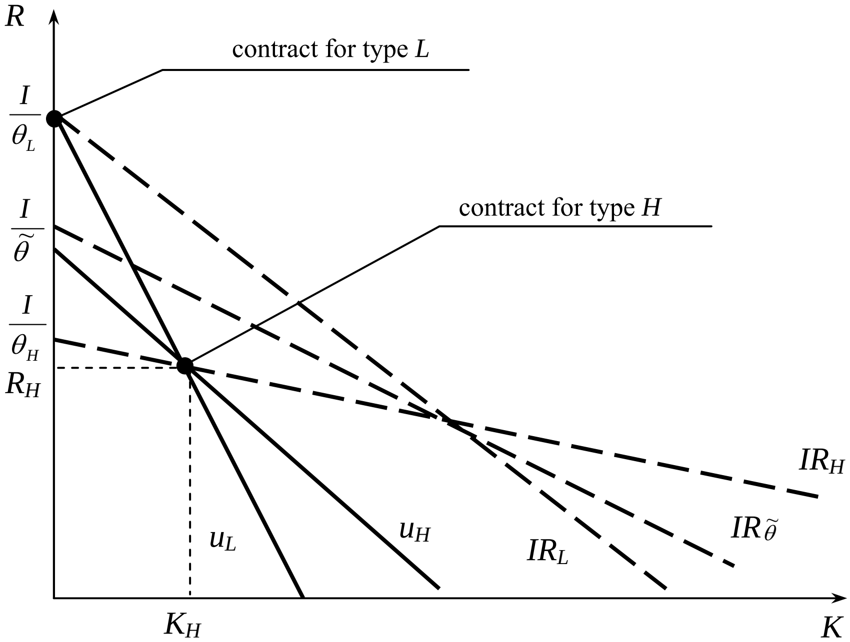
The only case when pooling equilibrium is possible is the second one. In this case if $R = \frac{I}{\tilde{\theta}}$ and $K = 0$ there is no profitable deviation for the agents.

Separating equilibrium: in this case contract for each type lies on the corresponding IR constraint:

$$\theta_L R + (1-\theta_L)xK - I = 0 \text{ for type } L \qquad (IR_L)$$

$$\theta_H R + (1-\theta_H)xK - I = 0 \text{ for type } H \qquad (IR_H)$$

The contract which delivers maximum utility for type L agent and still satisfies investor's individual rationality constraint is $R = \frac{I}{\theta_L}$, $K = 0$. In this case type H agent can offer contract (K_H, R_H) which strictly increases her utility and satisfies investor's IR constraint.



The contract (K_H, R_H) can be found as a solution of the system

$$\theta_L(C - R_H) - (1 - \theta_L)K_H = \theta_L(C - \frac{I}{\theta_L}),$$

$$\theta_H R_H + (1 - \theta_H)xK_H - I = 0.$$

PLEASE SOLVE

Problem 11.

Agent's utility:

$$U_A = (1 - p(a))u(W_0 + R(0)) + p(a)\int_0^\infty u(W_0 - x + R(x))g(x)dx - \psi(a) \to \max_{\{a, R(x)\}}$$

Insurer's profit:

$$U_I = -(1 - p(a))R(0) - p(a)\int_0^\infty R(x)g(x)dx$$

First-best: maximize agent's utility under insurer's individual rationality constraint

$$U_A \to \max_{\{a, R(x)\}}$$

$$\text{s.t. } U_I \geq 0.$$

Lagrange function:

$$L = (1 - p(a))u(W_0 + R(0)) + p(a)\int_0^\infty u(W_0 - x + R(x))g(x)dx - \psi(a) -$$

$$-\lambda[(1 - p(a))R(0) + p(a)\int_0^\infty R(x)g(x)dx] \to \max_{\{a, R(x)\}}$$

$$\frac{\partial L}{\partial a} = -p'(a)u(W_0 + R(0)) + p'(a)\int_0^\infty u(W_0 - x + R(x))g(x)dx - \psi'(a) -$$

$$-\lambda[-p'(a)R(0) + p'(a)\int_0^\infty R(x)g(x)dx] = 0, \quad (1)$$

$$\frac{\partial L}{\partial R(0)} = (1 - p(a))u'(W_0 + R(0)) - \lambda(1 - p(a)) = 0, \quad (2)$$

$$\frac{\partial L}{\partial R(x)} = p(a)u(W_0 - x + R(x))g(x) - \lambda p(a)R(x)g(x) = 0. \quad (3)$$

From (2) $u'(W_0 + R(0)) = \lambda$, from (3) $u'(W_0 - x + R(x)) = \lambda$, then $\forall\ x > 0$ $u'(W_0 - x + R(x)) = u'(W_0 + R(0))$ => $W_0 - x + R(x) = W_0 + R(0)$ => $R(x) = x + R(0)$ (Full insurance).
Substituting $R(x)$ into IR constraint we get

$$(1 - p(a))R(0) + p(a)\int_0^\infty (x + R(0))g(x)dx = (1 - p(a))R(0) + p(a)R(0) + p(a)\int_0^\infty xg(x)dx =$$

$$= R(0) + p(a)E = 0,$$

where $E = E(x \mid x > 0)$.

Substituting $R(x)$ into (1) we get

$$\psi'(a) + \lambda p'(a)E = \psi'(a) + u'(W_0 + R(0))p'(a)E = 0.$$

Then $u'(W_0 + R(0))E = -\frac{\psi'(a)}{p'(a)}$.

So, in the first-best outcome the agent will be fully insured, the insurance schedule will be linear and the agent's payment $R(0)$ and the optimal efforts level a^* will be determined from the equations

$$R(0) + p(a^*)E = 0, \quad u'(W_0 + R(0))E = -\frac{\psi'(a^*)}{p'(a^*)}.$$

Second-best: the agent maximizes her utility under certain insurance schedule $R(x)$:

$$U_A = (1 - p(a))u(W_0 + R(0)) + p(a)\int_0^\infty u(W_0 - x + R(x))g(x)dx - \psi(a) \to \max_a$$

$$\frac{\partial U_A}{\partial a} = -p'(a)u(W_0 + R(0)) + p'(a)\int_0^\infty u(W_0 - x + R(x))g(x)dx - \psi'(a) = 0$$

Then

$$\int_0^\infty u(W_0 - x + R(x))g(x)dx - u(W_0 + R(0)) = \frac{\psi'(a)}{p'(a)} \text{ if } a > 0$$

$$\int_0^\infty u(W_0 - x + R(x))g(x)dx - u(W_0 + R(0)) \geq \frac{\psi'(0)}{p'(0)} \text{ if } a = 0 \text{ (since } p'(a) < 0)$$

Let's consider the full insurance contract: $R(x) = x + R(0)$. Then

$$\int_0^\infty u(W_0 - x + R(x))g(x)dx - u(W_0 + R(0)) = 0 \geq \frac{\psi'(0)}{p'(0)} \text{ since } \psi'(a) \geq 0, p'(a) < 0.$$

Hence under the full insurance the agent will not undertake any effort.
The agent's payment $R(0)$ can be determined from the insurer's individual rationality constraint: $R(0) = -p'(0)E$.
Using nonlinear insurance schedules is not optimal for the insurer since in this case he can increase his profit offering full insurance for the agent.

So in the second-best outcome the agent again will be fully insured, but her efforts level will be $a = 0$. **PROBABLY SECOND BEST IS IN BETWEEN – SOME BUT NOT PERFECT INSURANCE AND SOME POSITIVE EFFORT**

Problem 15.
a). Expected entrepreneur utility if implementing C_1:

$U_1 = 5(1 - a) + 45a - 40a^2 - 6 \to \max_{a \in A}$, where $A = \{0, ⅓, ½\}$ => $a_1^* = ½$., $U_1^* = 9$.

Expected entrepreneur utility if implementing C_2

$U_2 = 48a - 40a^2 - 6 \to \max_{a \in A}$, $A = \{0, ⅓, ½\}$ => $a_2^* = ½$, $U_2^* = 8$.

So entrepreneur will choose project C_1 and effort $a^* = ½$.

b). Debt financing. Under debt financing entrepreneur maximizes:
If implements C_1

$$U_1 = (5 - r_0(\tilde{a}))(1 - a) + (45 - r_1(\tilde{a}))a - 40a^2 \to \max_{a, \tilde{a}},$$

where $\tilde{a}$ is the efforts which she declares and a is the efforts which she implements.
The constraints for this problem are:
- creditor's individual rationality constraints

$$r_0(a)(1 - a) + r_1(a)\, a \geq I,\ \forall\ a \in A, \qquad \text{(IR)}$$

- bilateral limited liability constraints

$$0 \leq r_0(a) \leq 5,\ 0 \leq r_1(a) \leq 45,\ \forall\ a \in A. \qquad \text{(LL)}$$

Assuming, that for bad outcome limited liability constraint is binding we get:
$\tilde{a} = 0$ => IR is violated (no investment occurs in this case);
$\tilde{a} = ⅓$ => $r_0 = 5$, $r_1 = 8$;
$\tilde{a} = ½$ => $r_0 = 5$, $r_1 = 7$.
Then the entrepreneur's expected utility is

$\tilde{a}$ \ a	0	⅓	½	
0	–	–	–	IR violated
⅓	0	$^{71}/_{9}$	$^{17}/_{2}$	
½	0	$^{74}/_{9}$	9	

It reaches its maximum if $a = \tilde{a} = ½$.

If entrepreneur implements C_2 she maximizes

$$U_2 = -r_0(\tilde{a})(1 - a) + (48 - r_1(\tilde{a}))a - 40a^2 \rightarrow \max_{a,\tilde{a}},$$

under the same creditor's individual rationality constraints

$$r_0(a)(1 - a) + r_1(a)\, a \geq I,\ \forall\ a \in A, \qquad \text{(IR)}$$

and bilateral limited liability constraints having the form

$$r_0(a) = 0,\ 0 \leq r_1(a) \leq 48,\ \forall\ a \in A. \qquad \text{(LL)}$$

From the IR constraint one can get:
$\tilde{a} = 0$ => IR is violated (no investment occurs in this case);
$\tilde{a} = ⅓$ => $r_0 = 0$, $r_1 = 18$;
$\tilde{a} = ½$ => $r_0 = 0$, $r_1 = 12$.
Then the entrepreneur's expected utility is

$\tilde{a}$ \ a	0	⅓	½	
0	–	–	–	IR violated
⅓	0	$^{50}/_{9}$	5	
½	0	$^{68}/_{9}$	8	

It again reaches its maximum if $a = \tilde{a} = ½$.

So, the first-best can be achieved in this case.

c). Equity financing. Under equity financing entrepreneur maximizes:
If implements C_1

$$U_1 = (1 - s(\tilde{a}))(5(1 - a) + 45a) - 40a^2 \rightarrow \max_{a,\tilde{a}},$$

where $\tilde{a}$ is the efforts which she declares and a is the efforts which she implements.
The constraints for this problem are:
- investor's individual rationality constraints

$$s(a)(5(1-a)+45\,a) \geq I,\ \forall\ a \in A, \qquad \text{(IR)}$$

- bilateral limited liability constraints

$$0 \leq s(a) \leq 1,\ \forall\ a \in A. \qquad \text{(LL)}$$

Then we get:

$\tilde{a} = 0$ => LL is violated (no investment occurs in this case);
$\tilde{a} = 1/3$ => $s = 18/55$;
$\tilde{a} = 1/2$ => $s = 6/25$.

Then the entrepreneur's expected utility is

$\tilde{a} \setminus a$	0	⅓	½	
0	–	–	–	LL violated
⅓	0	$71/9$	$75/11$	
½	0	$427/45$	9	

It is profitable for the entrepreneur to implement a = ⅓ regardless of her claim $\tilde{a}$. In the case $\tilde{a}$ = ½ the investor will have negative expected profit, so the investment will not occur. So, the investment occurs only in the case $\tilde{a}$ = ⅓ and the entrepreneur gets $U_1 = 71/9$.

If entrepreneur implements C_2 she maximizes

$$U_2 = 48a(1-s(\tilde{a})) - 40a^2 \rightarrow \max_{a,\tilde{a}},$$

under investor's individual rationality constraints

$$48\,a\,s(a) \geq I,\ \forall\ a \in A, \qquad \text{(IR)}$$

and bilateral limited liability constraints

$$0 \leq s(a) \leq 1,\ \forall\ a \in A. \qquad \text{(LL)}$$

Then:

$\tilde{a} = 0$ => IR is violated (no investment occurs in this case);
$\tilde{a} = 1/3$ => $s = 3/8$;
$\tilde{a} = 1/2$ => $s = 1/4$.

Then the entrepreneur's expected utility is

$\tilde{a} \setminus a$	0	⅓	½	
0	–	–	–	IR violated
⅓	0	$50/9$	5	
½	0	$68/9$	8	

This function reaches its maximum when $a = \tilde{a}$ = ½., $U_2^* = 8$.

So, the entrepreneur will choose C_2 in this case => FB cannot be implemented under equity financing.

Contracts HA #3 solutions

41.

a). First-best

$$v(x) - x \to \max \Rightarrow \text{first-best investment } x^*: v'(x^*) = 1$$

Ex-post spot contracting: parties wait for $t = 1$ and negotiate the surplus sharing.
Upstream-supplier utility:

$$\tfrac{1}{2} v(x) - x \to \max \Rightarrow v'(x) = 2$$

Since v' is decreasing => $x < x^*$.

b). $V(x) = f(v(x))$, $f' \geq 0$, $f'' \leq 0$.
First-best

$$V(x) - x \to \max \Rightarrow V'(x^0) = f'(v(x^0))v'(x^0) = 1$$

if $f'(v(x^0)) < 1$ then $v'(x^0) > 1 \Rightarrow x^0 < x^*$.

c). Consider two property rights allocations:
A: either upstream or downstream producer owns the computer;
B: a third-party owner owns the computer.

A. Upstream-supplier maximizes

$$\tfrac{1}{2} V(x) - x \to \max \Rightarrow V'(x^A) = f'(v(x^A))v'(x^A) = 2$$

B. Upstream-supplier maximizes

$$\tfrac{1}{2} v(x) - x \to \max \Rightarrow v'(x^B) = 2$$

Since $f'' \leq 0$, $v'' < 0 \Rightarrow (f'v')' = f''v' + f'v'' \leq 0 \Rightarrow (f'v')$ – decreasing => $x^A < x^0$ always => in the case A upstream-supplier always underinvests.

If $f' > 1 \Rightarrow v'(x^A) = \dfrac{2}{f'(v(x^A))} < 2 = v'(x^B) \Rightarrow x^A > x^B$, i.e. in the case A upstream-supplier underinvests less.

If $\tfrac{1}{2} < f' < 1 \Rightarrow v'(x^0) = \dfrac{1}{f'(v(x^0))} < 2 = v'(x^B) \Rightarrow x^0 > x^B$ => in the case B upstream-supplier underinvests.

$v'(x^A) = \dfrac{2}{f'(v(x^A))} > 2 = v'(x^B) \Rightarrow x^A < x^B$, then $x^0 > x^B > x^A$ => allocation B dominates A

d). If $f' < \tfrac{1}{2} \Rightarrow v'(x^0) = \dfrac{1}{f'(v(x^0))} > 2 = v'(x^B) \Rightarrow x^0 < x^B$ => in the case B upstream-supplier makes excessive investment.
But in the case A she underinvests (see **c**), then $x^B > x^0 > x^A$ and we cannot say any definite about the social welfare in these cases.

e). Let's calculate Shapley value for property rights allocations A and B:

K	$v(K)$	
	A	**B**
1	0	0
2	0	0
3	0	0
1, 2	$V(x^1)$	$v(x^2)$
1, 3	0	0
2, 3	0	0
1, 2, 3	$V(x^1)$	$V(x^2)$
Shapley value	$(½V(x^1), ½V(x^1), 0)$	$({}^1/_6v(x^2) + {}^1/_3V(x^2),$ ${}^1/_6v(x^2) + {}^1/_3V(x^2),$ ${}^1/_3V(x^2) – {}^1/_3v(x^2))$

where $x^1 = \arg\max(½V(x) – x)$, $x^2 = \arg\max({}^1/_6v(x) + {}^1/_3V(x) – x)$

Hence $x^1 = x^A$, and x^2 is the solution of

$$^1/_6v'(x^2) + {}^1/_3 f'(v(x^2))v'(x^2) = 1 \Rightarrow v'(x^2)(1 + 2f'(v(x^2))) = 6.$$

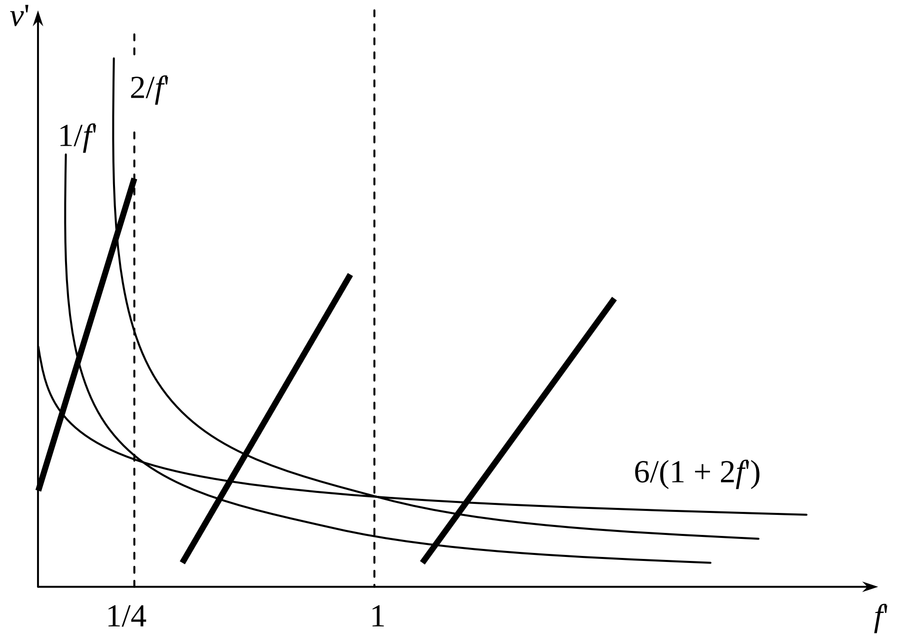
Now one have to compare:

$x^0 : v'(x^0) = \dfrac{1}{f\,'(v(x^0))}$

$x^1 : v'(x^1) = \dfrac{2}{f\,'(v(x^1))}$

$x^2 : v'(x^2) = \dfrac{6}{1+2f\,'(v(x^2))}$

Let's draw them in the plane (f', v'):



Then if $f' > 1 \Rightarrow x^0 > x^A > x^B \Rightarrow$ A dominates B.
If $1/4 < f' < 1 \Rightarrow x^0 > x^B > x^A \Rightarrow$ B dominates A.
If $f' < 1/4 \Rightarrow x^B > x^0 > x^A \Rightarrow$ we cannot say anything certain about the dominance.

42.

a). First-best:

$$W = R(x_1) - C(x_2) - x_1 - x_2 \to \max$$

Then from the first-order conditions

$$R'(x_1) = 1, \quad C'(x_2) = -1.$$

First-best solution does not depend on the assets allocation.

b). Outside options of the parties $d = (r(x_1, A_1) - P_m, P_m - c(x_2, A_2))$, surplus which they divide is

$$S = R(x_1) - C(x_2) - r(x_1, A_1) + c(x_2, A_2).$$

Ex-ante agents utilities

$u_1(x_1, x_2) = ½\, S + d_1 - x_1 = ½\, [R(x_1) - C(x_2) - r(x_1, A_1) + c(x_2, A_2)] + r(x_1, A_1) - P_m - x_1$,

$u_2(x_1, x_2) = ½\, S + d_2 - x_2 = ½\, [R(x_1) - C(x_2) - r(x_1, A_1) + c(x_2, A_2)] + P_m - c(x_2, A_2) - x_2$.

The solution of their problems is

$R'(x_1) + r'(x_1, A_1) = 2$,

$-C'(x_2) - c'(x_2, A_2) = 2$.

Since

$R'(x_1) > r'(x_1, \{a_1, a_2\}) \geq r'(x_1, \{a_i\}) \geq r'(x_1, \varnothing)$,

$-C'(x_2) > -c'(x_2, \{a_1, a_2\}) \geq -c'(x_2, \{a_i\}) \geq -c'(x_2, \varnothing)$,

the investment levels under different property rights allocations will be the following:

$x_1(\varnothing) \leq x_1(\{a_i\}) \leq x_1(\{a_1, a_2\}) < x_1^{FB}$,

$x_2(\varnothing) \leq x_2(\{a_i\}) \leq x_2(\{a_1, a_2\}) < x_2^{FB}$.

Nonintegration is optimal if

$$W(x_1(\{a_1\}), x_2(\{a_2\})) \geq \max\{ W(x_1(\varnothing), x_2(\{a_1, a_2\})), W(x_1(\{a_1, a_2\}), x_2(\varnothing))\}$$

Without knowing the form of the functions R, C, r and c it is difficult to identify the conditions required. I can point out only special cases, e.g. if $r'(x_1, \{a_1, a_2\}) = r'(x_1, \{a_i\}) > r'(x_1, \varnothing)$ and $c'(x_2, \{a_1, a_2\}) = c'(x_2, \{a_i\}) < c'(x_2, \varnothing)$ then both x_i under nonintegration are not less than under either agent integration => nonintegration is surely optimal.

c). Manager 1 integration is optimal, e.g. if $r'(x_1, \{a_1, a_2\}) > r'(x_1, \{a_i\})$ and $c'(x_2, \{a_1, a_2\}) = c'(x_2, \{a_i\}) = c'(x_2, \varnothing)$.

d). Ex-ante agents utilities

$u_1(x_1, x_2) = R(x_1) - C(x_2) - P_m + c(x_2, A_2) - x_1$,

$u_2(x_1, x_2) = P_m - c(x_2, A_2) - x_2$.

The solution is

$R'(x_1) = 1 \Rightarrow x_1(A_1) = x_1^{FB}$,

$-c'(x_2, A_2) = 1 \Rightarrow x_2(\varnothing) \le x_2(\{a_i\}) \le x_2(\{a_1, a_2\}) < x_2^{FB}$.

So, the optimal property rights allocation is manager 2 integration.

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43. a). Utility functions

$$U_{\text{firm}}(R(\theta), L(\theta), P(\theta)) = \sum_{\theta\in\Theta} p(\theta)[C_1(\theta) + (1 - \frac{L(R(\theta))}{A})C_2(\theta) - P(R(\theta))] \to \max_{\{R(\theta)\}}$$

$$U_{\text{inv}}(R(\theta), L(\theta), P(\theta)) = \sum_{\theta\in\Theta} p(\theta)[P(R(\theta)) + L(R(\theta))] - I \to \max_{\{P(\theta), L(\theta)\}}$$

where $p(\theta)$ – probability of the state, $R(\theta)$ – state, declared by firm.

Contracting problem: maximize U_{firm} under constraints:

$P(\theta) \le C_1(\theta)$, $0 \le L(\theta) \le 1$;

IC: $C_1(\theta) + (1 - \frac{L(\theta)}{A})C_2(\theta) - P(\theta) \ge C_1(\theta) + (1 - \frac{L(\tilde{\theta})}{A})C_2(\theta) - P(\tilde{\theta})$, $\forall\, \theta, \tilde{\theta}$;

IR: $U_{\text{inv}}(\theta, L(\theta), P(\theta)) \ge 0$.

b). First-best solution: since both parties are risk-neutral we can find it solving

$$U_{\text{firm}} + U_{\text{inv}} = \sum_{\theta\in\Theta} p(\theta)[C_1(\theta) + C_2(\theta) - (\frac{C_2(\theta)}{A} - 1)L(\theta)] - I \to \max_{\{L(\theta)\}} \Rightarrow L(\theta) \equiv 0.$$

Since $\sum_{\theta\in\Theta} p(\theta)[C_1(\theta) + C_2(\theta)] \ge \beta C_1^H \ge I$ investment is always made.

c). It follows from **IC** that:

$$C_1^H + (1 - \frac{L^{HH}}{A})C_2^H - P^{HH} \ge C_1^H + (1 - \frac{L^{HL}}{A})C_2^H - P^{HL},$$

$$C_1^H + (1 - \frac{L^{HL}}{A})C_2^L - P^{HL} \ge C_1^H + (1 - \frac{L^{HH}}{A})C_2^L - P^{HH}.$$

Subtracting the latter from the former one can get

$$\frac{1}{A}(L^{HL} - L^{HH})(C_2^H - C_2^L) \ge 0 \Rightarrow L^{HL} \ge L^{HH}.$$

The second inequality can be re-written in the form:

$$P^{HH} - P^{HL} \ge \frac{1}{A}(L^{HL} - L^{HH})C_2^L \Rightarrow P^{HH} \ge P^{HL}.$$

From the constraint $P(\theta) \le C_1(\theta)$ => $P^{LH} = P^{LL} = 0$.
Then incentive compatibility constraints for the case when $C_1 = 0$ give

$$(1-\frac{L^{LH}}{A})C_2^H \ge (1-\frac{L^{LL}}{A})C_2^H,$$

$$(1-\frac{L^{LL}}{A})C_2^L \ge (1-\frac{L^{LH}}{A})C_2^L.$$

These equations are satisfied only if $L^{LL} = L^{LH}$.
So, the second-best optimal contract looks like

$P^{HH} \ge P^{HL} \ge 0$, $P^{LH} = P^{LL} = 0$, $L^{HL} \ge L^{HH}$, $L^{LL} = L^{LH}$.

d). Contracting problem:

$$\text{U}_{\text{firm}} = (1-\beta)[(1-\frac{L}{A})(\gamma C_2^L + (1-\gamma)C_2^H)] + \beta[C_1^H + \gamma(1-\frac{L^{HH}}{A})C_2^H +$$
$$+(1-\gamma)(1-\frac{L^{HL}}{A})C_2^L - \gamma P^{HH} - (1-\gamma)P^{HL}] \to \max$$

IC: $\frac{1}{A}(L - L^{HH})C_2^H \ge P^{HH}$ (firm will not declare state (L,*) in the state (H,H));

$\frac{1}{A}(L - L^{HL})C_2^L \ge P^{HL}$ (firm will not declare (L,*) in the state (H,L));

$P^{HH} - P^{HL} \ge \frac{1}{A}(L^{HL} - L^{HH})C_2^L$ (firm will not declare (H,H) in the state (H,L));

$P^{HH} - P^{HL} \le \frac{1}{A}(L^{HL} - L^{HH})C_2^H$ (firm will not declare (H,L) in the state (H,H));

IR: $\beta\gamma(P^{HL} + L^{HL}) + \beta(1-\gamma)(P^{HH} + L^{HH}) + (1-\beta)L = I$;

$P^{HH} \le C_1^H$, $P^{HL} \le C_1^H$, $0 \le L, L^{HH}, L^{HL} \le 1$.

U_{firm} is decreasing in all of its arguments, so we should take minimal values. Intuitively in the good states it is unprofitable to liquidate the assets, so let's take $L^{HH} = 0$.
Since $C_2^H > A$ then the coefficient at L^{HL} is greater in absolute value than the one at P^{HL} => first we should minimize L^{HL}. This implies that

$$P^{HH} - P^{HL} = \frac{1}{A}L^{HL}C_2^H \Rightarrow P^{HH} - P^{HL} > \frac{1}{A}L^{HL}C_2^L$$

But then we can decrease both P^{HH} and L^{HL} while **IR** is satisfied => let's check $P^{HH} = P^{HL} = P$ and $L^{HL} = L^{HH} = 0$. In this case the first and the second IC reduce to

$LC_2^H > AP$ and $LC_2^L = AP$.

The latter equation together with **IR** gives

$$L = \frac{AI}{\beta(C_2^L - A) + A}, \quad P = \frac{IC_2^L}{\beta(C_2^L - A) + A}.$$

We should check whether this solution satisfies $P \le C_1^H$, $L \le 1$
If yes, then this is optimal contract. Since in a good state the assets are not liquidated => we can achieve first-best.
In the other case we should make the corresponding constraint binding and solve for L^{H*}. In this case the assets will be liquidated in the good state => the solution ex-post inefficient.

c). In a good state the firm maximizes:

$$\text{U}_{\text{firm}} = (1 - \frac{L^{H*}}{A})C_2^{H*} - P$$

Investor's IR constraint now is

$$P^{H*} + L^{H*} \geq I \ .$$

Since the firm has full bargaining power then the renegotiation on this stage implies that

$$P^{H*} = I < C_1^{\ H}, L^{H*} = 0.$$

Then ex-ante investor individual rationality constraint now is

$$\beta I + (1 - \beta)L \geq I \Rightarrow L \geq I.$$

Optimal renegotiation-proof contract is $P^{H*} = I$, $L^{H*} = 0$, $P^{L*} = 0$, $L^{L*} = I$.
So, the investment will take place only if the size of the assets at $t = 1$ $A \geq I$.

New Economic School

Contract Theory Final Exam 2005 Solutions

1. Consider an economy with a continuum of agents who produce output q by supplying input a (for effort) with the individual production function $q = \theta a$, where θ is an idiosyncratic productivity parameter. The productivity density in the population is given by $f(\theta)$ with support $[\underline{\theta}, \bar{\theta}]$, where $f(\theta) > \varepsilon > 0$. All agents have the same utility function $u(c) - a$ with $u' > 0$ and $u'' < 0$, where c is consumption.

a. What is the distribution of output and consumption in the economy when each agent lives and works in autarchy?

b. Suppose that all agents in this economy can write an insurance contract on a competitive insurance market before they know their productivity type. All agents are identical ex ante, and their future productivity is i.i.d. with $f(\theta)$. What is the optimal insurance contract when θ and a are observable ex post? What is the optimal insurance contract when only θ is observable? What is the optimal contract when neither θ nor a is observable ex post and $f(\theta)/(1 - F(\theta))$ is monotonically increasing?

c. Interpret the last solution. Show that the marginal premium is given by $P'(\theta) = (\theta u'(c(\theta)) - 1)c'(\theta)$ and $P'(\underline{\theta}) = P'(\bar{\theta}) = 0$. Discuss the solution.

a. $W = u(c) - a \to$ max; s.t. $c = q = \theta a \Rightarrow u'(q) = \dfrac{1}{\theta}$, $a^* = \dfrac{1}{\theta}(u')^{-1}(\dfrac{1}{\theta})$, $c^*(\theta) = \theta a = (u')^{-1}(\dfrac{1}{\theta})$.

b. If both θ and a are known to the insurance company then the contract has the form $P(a, \theta)$. Insurance company's problem is

$$\int_{\underline{\theta}}^{\bar{\theta}} P(a(\theta),\theta) f(\theta) d\theta \to \min;$$

s.t. $a(\theta) = \arg\max_{a\geq 0} (u(\theta a + P(a,\theta)) - a)$, **IC$_\theta$**

$$\int_{\underline{\theta}}^{\bar{\theta}} [u(\theta a + P(a(\theta),\theta)) - a(\theta)] f(\theta) d\theta \geq \int_{\underline{\theta}}^{\bar{\theta}} W(\theta) f(\theta) d\theta = R, \quad \textbf{IR}$$

where $W(\theta)$ is agent's of type θ maximum welfare in autarchy.

The solution of the agent's problem is $a(\theta) = \max\{\dfrac{1}{\theta}[(u')^{-1}(\dfrac{1}{\theta + P'(a(\theta),\theta)}) - P(a(\theta),\theta)], 0\}$

(it is only for the case when $P'(a(\theta), \theta)$ exists).

Substituting it into the insurance company criterion and solving this problem (I don't know how) one can find the optimal contract.

If only θ is known then insurance company's problem is

$$\int_{\underline{\theta}}^{\bar{\theta}} P(\theta) f(\theta) d\theta \to \min;$$

s.t. $a(\theta) = \arg\max_{a\geq 0} (u(\theta a + P(\theta)) - a)$,

$$\int_{\underline{\theta}}^{\bar{\theta}} [u(\theta a + P(\theta)) - a(\theta)] f(\theta) d\theta \geq R,$$

Agent's incentive compatibility gives $a = \max\{\dfrac{1}{\theta}[(u')^{-1}(\dfrac{1}{\theta}) - P(\theta)], 0\}$, $c = c^*(\theta) \Rightarrow$ his consumption does not depend on P, but only on effort choice. Then if θ is small enough the agent will not work at all, so the agent's expected utility is

$$\int_{\underline{\theta}}^{\tilde{\theta}}[u(P(\theta))]f(\theta)d\theta+\int_{\tilde{\theta}}^{\bar{\theta}}[u((u')^{-1}(\frac{1}{\theta}))-\frac{1}{\theta}((u')^{-1}(\frac{1}{\theta})-P(\theta))]f(\theta)d\theta \; =$$

$$= \int_{\underline{\theta}}^{\tilde{\theta}}[u(P(\theta))]f(\theta)d\theta+\int_{\tilde{\theta}}^{\bar{\theta}}[W(\theta)+\frac{P(\theta)}{\theta}]f(\theta)d\theta .$$

Lagrange function for the insurance company problem

$$L = \int_{\underline{\theta}}^{\bar{\theta}} P(\theta)f(\theta)d\theta \; + \lambda \; [\int_{\underline{\theta}}^{\tilde{\theta}}[u(P(\theta))]f(\theta)d\theta+\int_{\tilde{\theta}}^{\bar{\theta}}[W(\theta)+\frac{P(\theta)}{\theta}]f(\theta)d\theta-R] \; =$$

$$= \int_{\underline{\theta}}^{\tilde{\theta}}[\lambda u(P(\theta))+P(\theta)]f(\theta)d\theta+\int_{\tilde{\theta}}^{\bar{\theta}}(\frac{\lambda}{\theta}+1)P(\theta)f(\theta)d\theta+F \; ,$$

where F does not depend on $P(\theta)$.

We can maximize the terms separately:

- the first one gives $P(\theta)=(u')^{-1}(-\lambda)$, $\theta\in\left[\underline{\theta},\tilde{\theta}\right]$;
- the second one is linear in $P(\theta)$ => $P(\theta) = \begin{cases} P_{\max}, & \theta<\theta^* \\ P_{\min}, & \theta>\theta^* \end{cases}$, $\theta\in\left[\tilde{\theta},\bar{\theta}\right]$,

Since in the interval $\theta\in\left[\tilde{\theta},\bar{\theta}\right]$ $a > 0 \Rightarrow \theta^* \le \tilde{\theta} \Rightarrow P(\theta) = \begin{cases} (u')^{-1}(\lambda), & \theta<\tilde{\theta} \\ P_{\min}, & \theta>\tilde{\theta} \end{cases}$.

Then the individual rationality constraint gives

$$(u')^{-1}(\lambda)\int_{\underline{\theta}}^{\tilde{\theta}} f(\theta)d\theta+P_{\min}\int_{\tilde{\theta}}^{\bar{\theta}} f(\theta)d\theta=0 .$$

In the latter case the insurance company problem is

$$\int_{\underline{\theta}}^{\bar{\theta}} P(\theta)f(\theta)d\theta\rightarrow \min ;$$

s.t. $(a, \theta) = \arg\max_{(\tilde{a},\tilde{\theta})} (u(\theta\tilde{a} + P(\tilde{\theta})) - \tilde{a})$,

$$\int_{\underline{\theta}}^{\bar{\theta}}[u(\theta a+P(\theta))-a(\theta)]f(\theta)d\theta\ge R .$$

Let's denote $c(\tilde{\theta}) = \theta\tilde{a} + P(\tilde{\theta})$. Then $\tilde{a}=\frac{1}{\theta}(c(\tilde{\theta})-P(\tilde{\theta}))$. So the agent of type θ maximizes

$$u(c(\tilde{\theta})) - \frac{1}{\theta}(c(\tilde{\theta})-P(\tilde{\theta}))\rightarrow \max_{c(\tilde{\theta}),\tilde{\theta}}.$$

Maximizing with respect to $\tilde{\theta}$ one can get

$$P'(\tilde{\theta}) = (1 - \theta u'(c(\tilde{\theta})))\, c'(\tilde{\theta}).$$

Maximizing with respect to $c(\tilde{\theta})$, one can get

$$c(\theta) = (u')^{-1}(\frac{1}{\theta})$$

Then the agent will tell the truth if

$$P'(\theta) = (1 - \theta u'(c(\theta)))\, c'(\theta), \qquad (*)$$

$$c(\theta) = (u')^{-1}(\frac{1}{\theta}).$$

By substitution of $c(\theta)$ into the equation for $P'(\theta)$ it can be reduced to $P'(\theta) = 0$.

So the optimal contract here is a piecewise-constant function. Since in the previous case we also have found the piecewise-constant optimal contract in the problem with less constraints $\Rightarrow$ this contract remains optimal here.

c. Already proven, see (*).

2. A firm has a project requiring an investment of 20 at $t = 0$ for a sure return of 30 at $t = 1$. There is no discounting. The investment cost has to be raised from the financial market. Assume that a new equity issue is proposed. Potential new investors are uncertain about the value of the firm's assets in the place: $A \in \{50, 100\}$ with Prob($A = 100$) = 0.1.

a. Suppose that investors believe that both types of firms invest. What fraction of the firm's equity has to be issued to new investors? What are the payoffs to existing shareholders if they undertake the project? Are these beliefs reasonable?

b. Suppose that investors believe that only bad firms issue new equity. Same questions.

c. Suppose now that shareholders commit at t = 0 to a wasteful advertising campaign at t = 1 after the project return is realized. The advertising expenditure is an irreversible action on the part of the firm that results in a drop in profits of K. The size of the expenditure is a choice variable. Can a good firm signal its type via such expenditures ? Discuss.

a. Expected value of the firm's assets
$EA = 0.9*50 + 0.1*100 = 55$
Expected value of the firm after investment $EA' = 75$.
So the investors want $R = I/EA' = {}^{4}/_{15}$.
Firm's share after implementing the project:
$S_{50} = 80*(1 - R) = 58^{2}/_{3} > 50 \Rightarrow$ firm invests
$S_{100} = 130*(1 - R) = 95^{1}/_{3} < 100 \Rightarrow$ firm doesn't invest.
Investors' beliefs are not reasonable.

b. Investors' share $R = I/A'_{50} = {}^{2}/_{7}$.
Firm's share after implementing the project:
$S_{50} = 80*(1 - R) = 57^{1}/_{7} > 50 \Rightarrow$ firm invests
$S_{100} = 130*(1 - R) = 92^{6}/_{7} < 100 \Rightarrow$ firm doesn't invest.
Investors' beliefs are reasonable.

c. If investor distinguish firms' types he will require $r_{50} = {}^{2}/_{7}$ and $r_{100} = {}^{1}/_{6}$.

If the firm pays the cost of advertising out of its own pocket (e.g. before selling shares):

Firm's profit after the advertising:
$S_\theta = (1 - r_\theta)(\theta + 30) - K_\theta$.
Then the good firm can signal its type with K if:
IC₅₀: $S_{50} = 80*\,{}^{5}/_{7} - K_{50} \geq 80*\,{}^{5}/_{6} - K_{100}$
IR₅₀: $S_{50} = 80*\,{}^{5}/_{7} - K_{50} \geq 50$
IC₁₀₀: $S_{100} = 130*{}^{5}/_{6} - K_{100} \geq 130*\,{}^{5}/_{7} - K_{50}$,

IR_{100}: $S_{100} = 130*^5/_6 - K_{100} \geq 100$.
If we assume $K_{50} = 0 \Rightarrow$ from **IC_{50}** $K_{100} \geq {}^{200}/_{21} \Rightarrow$ even under minimum $K_{100} = {}^{200}/_{21}$ **IR_{100}** does not hold $\Rightarrow$ good firms will not signal their type.

If the firm shares the advertising cost with outside shareholders:

Firm's profit after the advertising:
$S_\theta = (1 - r_\theta)(\theta + 30 - K_\theta)$.
The good firm can signal its type with K if:
IC_{50}: $S_{50} = (80 - K_{50})*\ ^5/_7 \geq (80 - K_{100})*\ ^5/_6$
IR_{50}: $S_{50} = (80 - K_{50})*\ ^5/_7 \geq 50$
IC_{100}: $S_{100} = (130 - K_{100})*^5/_6 \geq (130 - K_{50})*\ ^5/_7$,
IR_{100}: $S_{100} = (130 - K_{100})*^5/_6 \geq 100$.
Assume again $K_{50} = 0 \Rightarrow$ from **IC_{50}** $K_{100} \geq {}^{80}/_7 \Rightarrow$ again for every $K_{100} \geq {}^{80}/_7$ **IR_{100}** does not hold $\Rightarrow$ in this case good firms will not signal their type too.

3. Consider a principal-agent problem with three exogenous states of nature, θ_1, θ_2 and θ_3; two effort levels, a_L and a_H; and two output levels, distributed as follows as function of the state of nature and effort level:

State	θ_1	θ_2	θ_3
Probability	0.25	0.5	0.25
Output under a_H	18	18	1
Output under a_L	18	1	1

The agent implements effort before he learns about the state of nature. The principal is risk-neutral, while the agent has utility function $\sqrt{w}$ when receiving monetary compensation w, minus the cost of effort, which is normalized to 0 for a_L and to 0.1 for a_H. The agent's reservation expected utility is 0.1.

a. Derive the first-best contract.

b. Derive the second-best contract when only output levels are observable.

c. Assume that principal can buy for a price p an information system that allows the parties to verify whether state of nature θ_3 happened or not. What maximum price will he pay for this system ? Discuss.

a. FB-contract: a is observable

a_H: $U_P = 0.75*18 + 0.25*1 - w_H \rightarrow$ max, s.t. $\sqrt{w_H} - 0.1 \geq 0.1 \Rightarrow w_H = 0.04 \Rightarrow U_P = 13.71$.

a_L: $U_P = 0.25*18 + 0.75*1 - w_L \rightarrow$ max, s.t. $\sqrt{w_L} \geq 0.1 \Rightarrow w_L = 0.01 \Rightarrow U_P = 5.24$.

So it is optimal to implement a_H and to pay $w_H = 0.04$.

b. Second-best contract:

$U_P = 0.75*(18 - w_{18}) + 0.25*(1 - w_1) \rightarrow$ max,

s.t. **IC:** $0.75\sqrt{w_{18}} + 0.25\sqrt{w_1} - 0.1 \geq 0.25\sqrt{w_{18}} + 0.75\sqrt{w_1}$

IR: $0.75\sqrt{w_{18}} + 0.25\sqrt{w_1} - 0.1 \geq 0.1$

Then w_{18} = 0.0625, w_1 = 0.0025, U_P = 13.7025.

c. If principal gets such a system, he can verify whether the agent implements a_H in the state θ_2. So he can offer the contract (w_{18}, w_2, w_3), where w_{18} – payment when q = 18, w_2 – payment when q = 1 and the state is not θ_3 and w_3 is the payment when the state is θ_3. His utility is

$U_P = 0.75*(18 - w_{18}) + 0.25*(1 - w_3) - p$, if the agent implements a_H and

$U_P = 0.25*(18 - w_{18}) + 0.5*(1 - w_2) + 0.25*(1 - w_3) - p$, if the agent implements a_L.

Here p is the price of the system.

If he wants the agent to implement a_H he should choose (w_{18}, w_2, w_3), such that

$U_P = 0.75*(18 - w_{18}) + 0.25*(1 - w_3) - p \to \max$,

s.t. **IC:** $0.75\sqrt{w_{18}} + 0.25\sqrt{w_3} - 0.1 \geq 0.25\sqrt{w_{18}} + 0.5\sqrt{w_2} + 0.25\sqrt{w_3}$

IR: $0.75\sqrt{w_{18}} + 0.25\sqrt{w_3} - 0.1 \geq 0.1$

Assume $w_2 = 0 \Rightarrow w_{18} = w_3 = 0.04 \Rightarrow$ principal can implement FB in this case. His utility

$U_P = U_P^{FB} - p = 13.71 - p$

He will buy the system if $13.71 - p \geq 13.7025 \Rightarrow p \leq 0.0075$.

4. A buyer B and a seller S write a contract to trade q units of good. B's utility of consuming q units is $2\sigma^{1/4}\omega q$ while the S's cost of producing q units is $q^2/(4\sigma^{1/4})$. Here σ is S's specific investment, and ω is the state of nature uniformly distributed on [0,1]. The timing is as follows: at t=0 B and S sign a contract, at t=1 S invests, at t=2 parties renegotiate and trade.

a. Derive the first best level of trade and investment.

b. Assuming that the S has no bargaining power, find the equilibrium level of trade and investment if parties sign no contract at t=0.

c. Suppose that under the conditions of part (b) parties sign the following contract: "at t=2, S sells Q units to B, and B pays S P dollars". Find the equilibrium level of trade and investment. Find the contract {Q*,P*} that implements the first best. Compare Q* to the average first best level of trade; provide intuition for the difference.

d. Solve (c) assuming equal distribution of bargaining power between B and S. Compare (c) and (d).

d. FB: W = $\int_0^1 2\sigma^{1/4}\omega q(\omega) - \frac{q^2(\omega)}{4\sigma^{1/4}} d\omega - \sigma \to \max$.

Maximizing w.r.t. q(ω) under certain σ, and then w.r.t. σ one can get:

$$2\sigma^{1/4}\omega q - q^2/(4\sigma^{1/4}) \to \max_q \Rightarrow q = 4\sigma^{1/2}\omega.$$

Substituting this into W, we get $\sigma^{FB} = 1$.

b. If S has no bargaining power => at t = 2 $P = q^2/(4\sigma^{1/4})$ (B lives him no surplus) => at t=1 S's expected utility is $U_S = -\sigma$ => σ = 0, and no trade occurs in this case.

c. t = 2: $U_B = \max_{q,\, p}\{2\sigma^{1/4}\omega q - p \mid \text{s.t. } IR_S \text{ holds}\}$ if renegotiation takes place.

IR_S: $p - q^2/(4\sigma^{1/4}) - \sigma = P - Q^2/(4\sigma^{1/4}) - \sigma \Rightarrow p = q^2/(4\sigma^{1/4}) + P - Q^2/(4\sigma^{1/4})$.

Substituting into U_B we get

$2\sigma^{1/4}\omega q - (q^2/(4\sigma^{1/4}) + P - Q^2/(4\sigma^{1/4})) \to \max_q$.

Then $q = 4\sigma^{1/2}\omega$.

At t = 1 seller maximizes

$U_S = E(p - q^2/(4\sigma^{1/4})) - \sigma = P - Q^2/(4\sigma^{1/4}) - \sigma \to \max_\sigma$, (since he gets the same utility in each state) =>

$\sigma^{5/4} = Q^2/16 \Rightarrow Q^* = 4$ – in this case $\sigma = \sigma^{FB}$.

P can be determined from IR constraint for S at t=0:

IR_S: $P - (Q^*)^2/(4\sigma^{1/4}) - \sigma^{FB} = 0 \Rightarrow P = 5$

FB-contract is Q = 4, P = 5.

d. If the parties have equal bargaining power.

t = 2: Surplus $V = 2\sigma^{1/4}\omega q - q^2/(4\sigma^{1/4}) - 2\sigma^{1/4}\omega Q + Q^2/(4\sigma^{1/4})$

$U_B = ½ V + 2\sigma^{1/4}\omega Q - P \to \max_q \Rightarrow q = 4\sigma^{1/2}\omega$.

Then seller's utility at t=2 in the state ω is

$U_S(\omega) = ½ V + P - Q^2/(4\sigma^{1/4}) = 2\sigma^{3/4}\omega^2 - \sigma^{1/4}\omega Q - Q^2/(8\sigma^{1/4}) + P$.

Expected seller's utility at t=0 is

$EU_S = \int U_S(\omega)d\omega - \sigma = {}^2/_3\sigma^{3/4} - ½\,\sigma^{1/4}Q - Q^2/(8\sigma^{1/4}) + P - \sigma \to \max_\sigma$.

FOC for this problem

$$½\,\sigma^{-1/4} - {}^1/_8\sigma^{-3/4}Q + Q^2/(32\sigma^{-5/4}) = 1.$$

$\sigma = \sigma^{FB} = 1$ satisfies it if $Q = 2 + \sqrt{20}$.

The expected surplus at t=0 if σ = 1 and $Q = 2 + \sqrt{20}$ is

$EW = EU_B + EU_S = {}^{13}/_3 \Rightarrow EU_S = {}^2/_3\sigma^{3/4} - ½\,\sigma^{1/4}Q - Q^2/(8\sigma^{1/4}) + P - \sigma = EW/2 \Rightarrow$

$$P = {}^{13}/_2 + \sqrt{20}.$$

FB-contract in this case $Q = 2 + \sqrt{20}$, $P = {}^{13}/_2 + \sqrt{20}$.

5. An entrepreneur with no initial wealth has a project that requires an initial investment *K* and whose output can take two values $q \in \{0, 1\}$. The market interest rate is normalized to zero. The entrepreneur offers a financial contract to an investor. After the initial investment, both parties observe the realization of the state of the world $\theta \in \{B, G\}$, which, however, is not observable by a court and thus is not contractable. Instead a contract can be contingent on the realization of a binary signal *s*, which is verifiable in a court. The signal is distributed as follows: If $\theta = G$ then $s = 1$ with probability one. If $\theta = B$ then $s = 1$ with probability γ and $s = 0$ with probability 1 – γ. Assume that γ is sufficiently small but strictly positive.

In each state of the world, an action *a* has to be taken: $a \in \{S, C\}$, where *S* is interpreted as "stop" and *C* is interpreted as "continue". The probability of high output depends on the realized state of the world and on the action chosen: $\Pr(q = 1 \mid \theta, a) = a_\theta$. (Note that a_θ also express the expected monetary return of the project given action *a* and state θ). While the investor cares only about monetary returns, the entrepreneur also has a private nonmonetary benefit *h* from choosing *C* rather than *S*. Monetary and non-monetary returns satisfy the following inequalities:

$$C_G < S_G < C_G + h$$

$C_B + h < S_B$

$S_G - C_G < S_B - C_B$

Actions cannot be described in an ex ante contract. Instead, a contract specifies control rights, that is, it specifies which party has the right to choose the action. Besides, the contract specifies the entrepreneur's compensation as a function of the realized s and q. If the party in control chooses an action that is not Pareto optimal, the parties can try to renegotiate to an optimal outcome. Assume that the entrepreneur has all the bargaining power in renegotiation.

a. For each of the following control structures, find the values of K for which a contract implementing the first-best action choice is feasible:

i. entrepreneurial control;

ii. investor control;

iii. contingent control (E has control when s = 1, I has control when s = 0).

b. Under what conditions does each control structure dominate?

a. First-best:

$$a^{FB}(\theta) = \begin{cases} C, & \theta = G \\ S, & \theta = B \end{cases}.$$

Investment occurs when $K \le \min\{(C_G + h), S_B\}$.

Let $P_{q,s}$ be payment to the entrepreneur. Then entrepreneur's gain is

State	Action	Payoff
G	C	$C_G P_{11} + (1 - C_G)P_{01} + h$
	S	$S_G P_{11} + (1 - S_G)P_{01}$
B	C	$C_B(\gamma P_{11} + (1 - \gamma)P_{10}) + (1 - C_B)(\gamma P_{01} + (1 - \gamma)P_{00}) + h$
	S	$S_B(\gamma P_{11} + (1 - \gamma)P_{10}) + (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00})$

Investor's gain is

State	Action	Payoff
G	C	$C_G(1 - P_{11}) - (1 - C_G)P_{01}$
	S	$S_G(1 - P_{11}) - (1 - S_G)P_{01}$
B	C	$C_B(1 - \gamma P_{11} - (1 - \gamma)P_{10}) - (1 - C_B)(\gamma P_{01} + (1 - \gamma)P_{00})$
	S	$S_B(1 - \gamma P_{11} - (1 - \gamma)P_{10}) - (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00})$

i. Entrepreneurial control.

The contract that implements FB under entrepreneur's control must satisfy following conditions

$\mathbf{IR_{I,G}}$: $C_G(1 - P_{11}) - (1 - C_G)P_{01} \ge K$

$\mathbf{IR_{I,B}}$: $S_B(1 - \gamma P_{11} - (1 - \gamma)P_{10}) - (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00}) \ge K$

$\mathbf{IC_{E,G}}$: $(C_G - S_G)(P_{11} - P_{01}) + h \ge 0$

$\mathbf{IC_{E,B}}$: $(S_B - C_B)(\gamma(P_{11} - P_{01}) + (1 - \gamma)(P_{10} - P_{00})) - h \geq 0$.

$P_{q,s} \geq 0 \ \forall q, s$.

Assume that $P_{11} = P_{01} = 0$. Then $\mathbf{IR_{I,G}}$ is satisfied when $C_G \geq K$, $\mathbf{IC_{E,G}}$ is satisfied if $h \geq 0$.

To satisfy $\mathbf{IC_{E,B}}$ assume that $P_{00} = 0$, $P_{10} = \dfrac{h}{(1-\gamma)(S_B - C_B)}$.

Then $\mathbf{IR_{I,B}}$ is satisfied if : $S_B\left(1 - \dfrac{h}{S_B - C_B}\right) \geq K$.

So, the investment occurs if $K \leq \min\{C_G, S_B\left(1 - \dfrac{h}{S_B - C_B}\right)\}$

ii. Investor control.

C can be implemented under investor's control in the state G if following inequality holds

$C_G(1 - P_{11}) - (1 - C_G)P_{01} \geq S_G(1 - P_{11}) - (1 - S_G)P_{01} \Rightarrow (C_G - S_G)(1 - P_{11} + P_{01}) \geq 0$.

Since $C_G < S_G$ this implies that $P_{11} - P_{01} > 1$, but this contradicts with the investor **IR** constraint $\Rightarrow$ investor will choose S.

Renegotiation: entrepreneur pays to investor Δ in order to implement C. Since entrepreneur has full bargaining power then

$C_G(1 - P_{11}) - (1 - C_G)P_{01} + \Delta = S_G(1 - P_{11}) - (1 - S_G)P_{01} \Rightarrow (C_G - S_G)(1 - P_{11} + P_{01}) + \Delta = 0$.

Entrepreneur will renegotiate if

$\mathbf{IC_{E,G}}$ $C_GP_{11} + (1 - C_G)P_{01} + h - \Delta \geq S_GP_{11} + (1 - S_G)P_{01} \Rightarrow (C_G - S_G)(P_{11} + P_{01}) + h - \Delta \geq 0$,

$\mathbf{IR_{E,G}}$ $C_GP_{11} + (1 - C_G)P_{01} + h - \Delta \geq 0$.

Investment takes place if

$\mathbf{IR_{I,G}}$ $S_G(1 - P_{11}) - (1 - S_G)P_{01} \geq K$.

In the state B S is implemented if

$\mathbf{IC_{I,B}}$ $(S_B - C_B)(1 - \gamma(P_{11} - P_{01}) - (1 - \gamma)(P_{10} - P_{00})) \geq 0 \Rightarrow \gamma(P_{11} - P_{01}) + (1 - \gamma)(P_{10} - P_{00}) \leq 1$

Investment takes place if

$\mathbf{IR_{I,B}}$ $S_B(1 - \gamma P_{11} - (1 - \gamma)P_{10}) - (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00}) \geq K$

Also must be satisfied

$\mathbf{IR_{E,B}}$ $S_B(\gamma P_{11} + (1 - \gamma)P_{10}) + (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00}) \geq 0$.

So the system describing FB-implementing contract is

$(C_G - S_G)(1 - P_{11} + P_{01}) + \Delta = 0$.

$\mathbf{IC_{E,G}}$ $(C_G - S_G)(P_{11} + P_{01}) + h - \Delta \geq 0$,

$\mathbf{IR_{E,G}}$ $C_GP_{11} + (1 - C_G)P_{01} + h - \Delta \geq 0$.

$\mathbf{IR_{I,G}}$ $S_G(1 - P_{11}) - (1 - S_G)P_{01} \geq K$.

$\mathbf{IC_{I,B}}$ $\gamma(P_{11} - P_{01}) + (1 - \gamma)(P_{10} - P_{00}) \leq 1$

$\mathbf{IR_{I,B}}$ $S_B(1 - \gamma P_{11} - (1 - \gamma)P_{10}) - (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00}) \geq K$

$\mathbf{IR_{E,B}}$ $S_B(\gamma P_{11} + (1 - \gamma)P_{10}) + (1 - S_B)(\gamma P_{01} + (1 - \gamma)P_{00}) \geq 0$.

$P_{q,s} \geq 0 \ \forall q, s$.

Assume that $P_{11} = P_{01} = 0 \Rightarrow \Delta = S_G - C_G$. From **$IC_{E,G}$** and **$IR_{E,G}$** $\Delta \leq h$ – it is satisfied since $S_G < C_G + h$. Then from **$IR_{I,G}$** $S_G \geq K$.

The other inequalities will take the form

$IC_{I,B}$ $(1-\gamma)(P_{10} - P_{00}) \leq 1$

$IR_{I,B}$ $S_B(1-(1-\gamma)P_{10}) - (1-S_B)(1-\gamma)P_{00} \geq K$

$IR_{E,B}$ $S_BP_{10} + (1-S_B)P_{00} \geq 0$.

All of them are satisfied if we assume $P_{10} = P_{00} = 0$. Then from **$IR_{I,B}$** $K \leq S_B$.

So, the investment occurs if $K \leq \min\{S_G, S_B\}$

iii. Contingent control.

If s = 1 the entrepreneur chooses an action, then following equations must be satisfied:

- in the good state

$IC_{E,G}$: $(C_G - S_G)(P_{11} - P_{01}) + h \geq 0$

$IR_{I,G}$: $C_G(1-P_{11}) - (1-C_G)P_{01} \geq K$

- in the bad state

$IC_{E,B}$: $(S_B - C_B)(P_{11} - P_{01}) - h \geq 0$.

$IR_{I,B}$: $S_B(1-P_{11}) - (1-S_B)P_{01} \geq K$

If s = 0 the investor chooses an action, then:

$IC_{I,B}$ $P_{10} - P_{00} \leq 1$

$IR_{I,B}$ $S_B(1-P_{10}) - (1-S_B)P_{00} \geq K$

$IR_{E,B}$ $S_BP_{10} + (1-S_B)P_{00} \geq 0$.

Assume $P_{01} = 0 \Rightarrow$ **$IC_{E,B}$** is satisfied if $P_{11} = \frac{h}{S_B - C_B} \Rightarrow$ from **$IR_{I,G}$** : $C_G(1 - \frac{h}{S_B - C_B} \geq K$, and from **$IR_{I,B}$** : $S_B(1 - \frac{h}{S_B - C_B}) \geq K$.

If we assume $P_{10} = P_{00} = 0 \Rightarrow$ from **$IR_{I,B}$** $S_B \geq K$ – it is always satisfied if **$IR_{I,B}$** holds.

So, the investment occurs if $K \leq (1 - \frac{h}{S_B - C_B})\min\{C_G, S_B\}$

b. If the investment occurs then all the structures leads to the same total welfare. So we should look at the conditions when these structures allow investment.

FB: $K \leq \min\{(C_G + h), S_B\}$

i. $K \leq \min\{C_G, S_B(1 - \frac{h}{S_B - C_B})\}$

ii. $K \leq \min\{S_G, S_B\}$

iii. $K \leq (1 - \frac{h}{S_B - C_B})\min\{C_G, S_B\}$

(iii) is always dominated by (i), and (i) is always dominated by (ii), since $C_G < S_G$.

If we assume that ex ante probabilities of the good and the bad states are equal then the conditions for the investment to be made are:

FB: $K \leq 0.5(C_G + h + S_B)$

i. $K \leq 0.5(C_G + S_B (1 - \frac{h}{S_B - C_B}))$

ii. $K \leq 0.5(S_G + S_B)$

iii. $K \leq 0.5(1 - \frac{h}{S_B - C_B})(C_G + S_B)$

It is readily seen that again (iii) p (i) p (ii).