A Practitioner's Guide to Market Microstructure Invariance

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Market microstructure invariance is the hypothesis, first proposed by Kyle and Obizhaeva [2016a], that dollar risk transfer and dollar transaction costs are the same for all stocks when trades are converted to bets, calendar time is converted to business time, and return volatility is converted to dollar volatility.

A *bet* is a transaction of a certain quantity of a particular stock that is intended to produce an idiosyncratic gain based upon the beliefs of one or more investors. Bets are difficult to distill from trades because they may encompass a single trade or many trades executed intermittently over varying times spans.

Business time is a less challenging concept. It is equal to the expected calendar time between bets. If there are four bets expected to arrive within a unit of calendar time, business time equals 25% of calendar time. In other words, business time passes four times faster than calendar time. We can also think of the rate at which bets arrive as market velocity. Velocity therefore equals the reciprocal of business time.

Risk, within the context of market microstructure invariance, is not equal to the volatility of returns in business time. Rather, it is equal to the volatility of a dollar quantity in business time. Specifically, it equals the product of the price of the stock at the initiation of a bet, the quantity of shares traded, and the return volatility in business time arising from order flow imbalances (return volatility times the reciprocal of the square root of velocity).

The hypothesis that dollar risk transfer and dollar transaction costs are invariant functions, together with these three definitions, constitutes a sufficient foundation for constructing a comprehensive framework of market microstructure invariance. The purpose of this article is to present this framework in a transparent and accessible style without sacrificing rigor.

THE MATHEMATICAL FRAMEWORK[1]

The Invariance of Risk Transfer

We start by defining *expected betting volume* as velocity times the number of shares an investor expects to trade in one bet. Recall that velocity equals the expected arrival rate of bets, which is the reciprocal of business time:

$$V_B = \gamma \cdot E|Q|. \tag{1}$$

In Equation 1, V_B equals expected betting volume, γ equals velocity, and $E|Q|$ equals the absolute value of the expected number of shares in buy or sell orders in one bet.

We can also relate betting volume to trading volume by introducing a volume multiplier, ζ:

$$V_B = \frac{2}{\zeta} \cdot V_T. \tag{2}$$

If there are no intermediaries, the volume multiplier equals one. Each trade would match two bets, given that investors on both sides of the trade are betting. If there is a single intermediary for all bets, then the volume multiplier equals two, because this intermediary increases trading volume but not betting volume. If each bet has its own intermediaries who trade their risk among each other, then the volume multiplier equals three. If bets are distributed among multiple intermediaries, then the volume multiplier is equal to four or more. The volume multiplier serves to offset the multiplication effect intermediaries have on trades, recognizing that their activities do not increase actual betting.

Next, we define *betting volatility* as the fraction of return volatility attributable to order flow imbalances, as opposed to return volatility that is driven by the public releases of new information that is incorporated into prices without trading. Betting volatility reflects the market impact of betting:

$$\sigma_B = \psi \cdot \sigma_R \tag{3}$$

In Equation 3, σ_B equals betting volatility, ψ equals the volatility multiplier, and σ_R equals return volatility. As a first approximation, we assume that both the volume multiplier and the volatility multiplier are constant.

The distinction between trading and betting variables is not essential to our invariance arguments, but we retain this distinction to highlight the difference between variables used to formulate invariance and variables that we observe in the data.

We are now able to define *risk transfer* as the dollar value of betting volatility multiplied by the reciprocal of the square root of velocity. The dollar value of betting volatility equals betting volatility as shown in Equation 3 multiplied by the size of the bet, which is price multiplied by quantity. Betting volatility scales with the square root of velocity because we wish to measure volatility in business time rather than calendar time. Therefore,

$$I_B = P \cdot Q \cdot \sigma_B \cdot \gamma^{-\frac{1}{2}} \tag{4}$$

where I_B equals the dollar amount of the risk transferred by the bet per tick in business time, and P equals the price of a share of stock at the initiation of a bet. Market microstructure invariance hypothesizes that the distribution of risk transfer I_B is the same for all assets. Scaling bet sizes $Q \cdot P$ by volatility in business time makes them invariant.

Equation 4 implies that holding other attributes constant, bets with higher velocity involve a greater number of shares traded per bet. This result occurs because business-time volatility is lower in high-velocity stocks than in low-velocity stocks, assuming they both have the same calendar-day betting volatility. Thus, for invariance to hold, the number of shares traded per bet for a high-velocity stock must be greater than the number of shares traded per bet for a low-velocity stock by a factor equal to the square root of the ratio of their respective velocities—or equivalently, by a factor equal to the square root of the ratio of high-velocity business time to low-velocity business time.

This relationship between velocity and the quantity of shares traded is a critical assumption of the invariance hypothesis, but it is hardly arbitrary: It follows first from principles. Suppose that large stocks present better opportunity for profit than small stocks, for example, because large stocks attract more noise traders. This enhanced opportunity will also attract a greater number of informed traders to trade large stocks; hence, velocity will be greater for large stocks than for small ones.

As this process unfolds, large stocks become more efficiently priced. Just as betting volatility scales with the square root of velocity, so does the average distance between market price and fundamental value. Traders need to make the same dollar profit per bet in order to recoup their entry costs. Therefore, they need to increase the quantity of shares traded per bet by an amount that is proportional to the square root of velocity. As a result, the dollar risk transferred will remain the same.[2]

Although some attributes of betting are difficult to observe, we can infer them from trading behavior. For example, trading activity equals the product of return volatility and the dollar value of trading volume, which are both readily observable:

$$W_T = \sigma_R \cdot PV_T. \tag{5}$$

Let W_T denote the amount of aggregate risk transferred per day. Betting activity similarly equals the product of betting volatility and the dollar value of betting volume, but these values are not easy to observe:

$$W_B = \sigma_B \cdot PV_B. \tag{6}$$

Fortunately, we can map trading activity onto betting activity as a function of the volume multiplier and the volatility multiplier:

$$W_B = W_T \cdot \frac{2\Psi}{\zeta}. \tag{7}$$

By rearranging Equation 4 to obtain

$$Q = \gamma^{\frac{1}{2}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot I_B \tag{8}$$

and recalling from Equation 1 that betting volume equals velocity times the expected number of shares traded in one bet, we can express betting activity as a function of velocity and the expected risk transfer as follows:

$$W_B = \gamma^{\frac{3}{2}} \cdot E|I_B|. \tag{9}$$

We show more detail about the derivation of Equation 9 in the appendix. As we mentioned earlier, we assume here that the volatility multiplier and the volume multiplier remain constant.

The quantity $\gamma^{\frac{3}{2}}$ in Equation 9 has an intuitive interpretation. We have already shown that as a condition of invariance, keeping prices the same, the number of shares traded must be greater for a high-velocity stock than a low-velocity stock by a factor equal to the square root of the ratio of their respective velocities. The quantity $\gamma^{\frac{3}{2}}$ equals the factor by which betting activity is greater. This factor can be decomposed into two components: the factor $W_B^{\frac{2}{3}}$ by which velocity is greater and the factor $W_B^{\frac{1}{3}}$ by which the number of shares traded per bet is greater. This decomposition shows that, to the extent trading activity is greater for one stock than another, velocity increases twice as fast as bet size, as long as both stocks have the same betting volatility.

Betting activity, together with invariance, allows us to infer velocity, as well as the average number of shares traded per bet. Velocity can be expressed as

$$\gamma = W_B^{\frac{2}{3}} \cdot \left(E|I_B|\right)^{\frac{-2}{3}}. \tag{10}$$

This implies that γ is proportional to $W_B^{\frac{2}{3}}$, which we write as $\gamma \sim W_B^{\frac{2}{3}}$, because as shown earlier, the distribution of risk transfer is the same for all stocks when measured in units of business time. And the average number of shares traded per bet is equal to

$$E|Q| = W_B^{\frac{1}{3}} \cdot \frac{1}{P\sigma_B} \cdot \left(E|I_B|\right)^{\frac{2}{3}}. \tag{11}$$

Therefore,

$$E|Q| \sim W_B^{\frac{1}{3}} \cdot \frac{1}{P\sigma_B}$$

because $E|I_B|$ is also constant across stocks again due to the invariance of risk transfer from Equation 4. We show more detail about the derivations of Equation 10 and Equation 11 in the appendix.

We now use Equation 10 to obtain the entire distribution of the number of shares traded as a fraction of betting volume:

$$\frac{Q}{V_B} = W_B^{\frac{-2}{3}} \cdot \left(E|I_B|\right)^{\frac{-1}{3}} \cdot I_B. \tag{12}$$

It follows that

$$\frac{Q}{V_B} \sim W_B^{\frac{-2}{3}} \cdot I_B.$$

Again, we provide additional detail about this derivation in the appendix.

In Equations 10, 11, and 12, $\left(E|I_B|\right)^{\frac{-2}{3}}$ and $\left(E|I_B|\right)^{\frac{-1}{3}}$ are constants calibrated from the observed distribution of risk transfer.[3] Equation 12 shows that the distribution of risk transfer $\left(E|I_B|\right)^{\frac{-1}{3}} \cdot I_B$ is linked to the distribution of the number of shares traded per bet as a fraction of betting volume $\frac{Q}{V_B}$.

The stock-specific term $W_B^{\frac{-2}{3}}$ in Equation 12 scales that distribution. The stock-specific term $W_B^{\frac{-2}{3}}$ in Equation 10 scales velocity.

Thus far, we have presented the mathematical structure that underlies the invariance of risk transfer. Now we turn to transaction costs.

The Invariance of Transaction Costs

Market microstructure invariance also hypothesizes that there is an invariant transaction cost function, which we denote as $C_B(I_B)$. This means that the dollar cost of executing bets in low- and high-velocity stocks is the same when measured as a function of the dollar amount of risk transferred by the scaled bets I_B.

The hypothesis that transaction costs are invariant rests on the assumption that investors expect to pay the same research cost to generate bets of the same size. These bets are expected to generate the same gross and net profit and, therefore, incur the same transaction cost. Because investors cannot foresee how much risk their bets will transfer until they have completed their research, the average cost $C_B = E\{C_B(\tilde{I}_B)\}$ of executing bets, which is the average transaction cost across all bet sizes, is also the same and equal to the unconditional expectation of $C_B(I_B)$.

We express the percentage cost of executing a bet as the invariant transaction cost function (the dollar cost of transferring the bet's risk) divided by the dollar size of the bet. Equation 13 gives the stock-specific percentage cost of executing a bet:

$$C(Q) = \frac{C_B(I_B)}{PQ}. \tag{13}$$

We next introduce an illiquidity measure, which we define as the average cost C_B of executing an average bet $E|PQ|$, expressed as a fraction of the value of the average bet:

$$\frac{1}{L} = \frac{C_B}{E|PQ|}. \tag{14}$$

The quantity $1/L$ is the volume-weighted transaction cost. Using Equation 11, it can be shown that this illiquidity measure is proportional to betting volatility in business time $\sigma_B W_B^{\frac{-1}{3}}$:

$$\frac{1}{L} = \sigma_B W_B^{\frac{-1}{3}} \cdot \left[\left(E|I_B| \right)^{\frac{-1}{3}} \right]^2 \cdot C_B. \tag{15}$$

It follows that

$$\frac{1}{L} \sim \sigma_B W_B^{\frac{-1}{3}},$$

because both $\left[\left(E|I_B| \right)^{\frac{-1}{3}} \right]^2$ and C_B in Equation 15 are constant, given the invariance of risk transfer and transaction costs. From Equation 6, we can define illiquidity as

$$\frac{1}{L} \sim \left[\frac{PV_B}{\sigma_B^2} \right]^{\frac{-1}{3}}.$$

The invariance of risk transfer allows us to express the percentage cost of executing a particular bet Q as a function of the average cost of executing a bet:

$$C(Q) = \frac{C_B}{E|PQ|} \cdot f(I_B). \tag{16}$$

The derivations of Equations 15 and 16 are shown in the appendix. In Equation 16, the function $f(I_B)$ equals

$$\frac{C_B(I_B)}{C_B} \Big/ \frac{|I_B|}{E|I_B|}.$$

It gives the invariant average cost function for executing a bet that transfers an invariant quantity of risk I_B, in which the cost function and the risk transfer function are expressed relative to their mean values. For example, $f(2) = 10$ would say that a bet twice as large as an average bet costs ten times more.

Equation 16 reveals that the cost of executing a bet can be decomposed into the product of illiquidity $\frac{1}{L}$ and the invariant average cost function $f(I_B)$. Note that bet size Q appears not only in the denominator of the right-hand side of Equation 16 but also in its numerator, because risk transfer I_B as defined in Equation 4 represents the amount of risk transferred by that bet.

Equations 10 and 11 allow us to express illiquidity in terms of betting volatility, betting activity, and costs. Recall that $\left(E|I_B| \right)^{\frac{-2}{3}}$ and $\left(E|I_B| \right)^{\frac{-1}{3}}$ are constants calibrated from the observed distribution of risk transfer. We also calibrate $f(I_B)$, $C_B(I_B)$ and, therefore, C_B from the data.[4]

To summarize, the *dollar cost* of executing bets is the same for all stocks if these bets transfer the same amount of risk, I_B. However, the *percentage cost* of executing a bet, which we can measure as a stock-specific cost function, differs across low- and high-velocity stocks. Nonetheless, when we scale these percentage cost functions by the stock-specific illiquidity measure, they again are invariant to velocity.

We next illustrate these invariance hypotheses with numerical examples.

NUMERICAL EXAMPLES

Risk Transfer

We begin by considering two stocks: a low-velocity stock, for which there are 100 bets per day, and a high-velocity stock, for which there are 400 bets per day. Business time therefore equals 1.0% and 0.25% of daily calendar time, respectively, for these two stocks. We assume that the price of each stocks equals \$40. We also assume that return volatility equals 2.0% per day for both stocks.

We consider several points along the distribution of the quantity of each bet: the mean, the median, and two standard deviations above the mean. Let's first consider the *mean bet*. We assume that the quantity of shares traded per bet for the low-velocity stock equals 20,000 shares. Thus, the size of the mean bet $P \cdot Q$ in a low-velocity stock equals \$800,000. For invariance to hold, the quantity of shares traded per bet for a high-velocity stock must equal the quantity of shares traded in a low-velocity stock multiplied by a factor equal to the square root of the ratio of high-velocity business time to low-velocity business time. Therefore, the size of the mean bet in a high-velocity stock equals \$1,600,000 [$40 \times 20{,}000 \times (1.0\%/0.25\%)^{1/2}$] or 40,000 shares.

We assume that each bet has its own intermediary for both the low- and high-velocity stocks; thus, the volume multiplier equals 3.0. We also assume that 64% of daily return volatility arises from order flow imbalances; thus, the volatility multiplier equals 0.64.

We calculate *betting volume* in two ways: as the product of velocity and the quantity of shares traded in the mean bet (Equation 1), or as two divided by the volume multiplier and then multiplied by trading volume (Equation 2). Thus, betting volume equals 2,000,000 shares for a low-velocity stock ($100 \times 20{,}000$ or $2 \div 3 \times 3{,}000{,}000$) and 16,000,000 shares for a high-velocity stock ($400 \times 40{,}000$ or $2 \div 3 \times 24{,}000{,}000$).

Daily trading volume is estimated from the mean bet. It is equal to half the volume multiplier times the product of the velocity and the quantity of shares traded in the mean bet. We divide the volume multiplier by two to arrive at trading volume because investors on both sides of a trade are betting. Therefore, trading volume equals 3,000,000 shares per day for a low-velocity stock ($3.0 \div 2 \times 100 \times 20{,}000$) and 24,000,000 shares per day for a high-velocity stock ($3.0 \div 2 \times 400 \times 40{,}000$).

To arrive at *daily betting volatility*, we multiply daily return volatility by the volatility multiplier (Equation 3), which equals 1.28% for both the low-velocity and high-velocity stocks ($2.0\% \times 0.64$). To convert daily betting volatility to *betting volatility in business time*, we divide it by the square root of velocity, which equals 0.128% for the low-velocity stock ($1.28\% \div 100^{1/2}$) and 0.064% for the high-velocity stock ($1.28\% \div 400^{1/2}$).

Next, we calculate *trading activity* and *betting activity*. Trading activity equals the product of price, trading volume, and daily return volatility (Equation 5), which is \$2,400,000 for a low-velocity stock ($40 \times 3{,}000{,}000 \times 2.0\%$) and \$19,200,000 for a high-velocity stock ($40 \times 24{,}000{,}000 \times 2.0\%$). Betting activity equals the product of price, betting volume, and betting volatility (Equation 7), which is \$1,024,000 for a low-velocity stock ($40 \times 2{,}000{,}000 \times 1.28\%$) and \$8,192,000 for a high-velocity stock ($40 \times 16{,}000{,}000 \times 1.28\%$). We could also calculate betting activity by multiplying trading activity by twice the volatility multiplier and then dividing by the volume multiplier.

We now have the necessary values to calculate the dollar value of the risk transferred by betting, which we show to be the same for both the low- and high-velocity bets, as we hypothesized. It is equal to the product of the size of the bet and daily betting volatility, divided by the square root of velocity (Equation 4). Thus, risk transfer equals \$1,024 for the mean bet in both a low-velocity stock ($\$40 \times 20{,}000 \times 1.28\% \div 100^{1/2}$) and a high-velocity stock ($\$40 \times 40{,}000 \times 1.28\% \div 400^{1/2}$). This invariance prevails irrespective of differences in velocity because investors adjust the quantity of the shares traded in order to maintain expected dollar profits, and higher velocity leads to more efficient pricing.

The calculations we have shown thus far pertain to the mean bets in low- and high-velocity stocks. We compute the same values for the bets located at the median and for the bets located two standard deviations above the mean. The cross-sectional distribution of unsigned quantity is approximately lognormal with log-variance equal to about 2.50.[5] Thus, the median quantity approximately equals the mean quantity divided by $e^{2.50/2}$, and the quantity located two standard deviations above the mean equals the median quantity multiplied by $e^{+2\sqrt{2.50}}$. Again, the amount of risk that is transferred by betting is the same for low- and high-velocity stocks at these other locations along the distribution. We show that the scaled bet sizes are the same \$288 for the median bets and \$6,927 for the two-standard-deviation bets.

These calculations are summarized in Exhibit 1. For the mean bet, the median bet, and the two-standard-deviation bet, there are two columns with calculations for low- and high-velocity stocks, respectively.

We earlier showed that we can calculate betting activity as the product of velocity raised to the 3/2 power and the expected distribution of risk transfer (Equation 9). If we raise the factor by which velocity is greater (in our example, 4) to the 3/2 power, we get the factor by which betting volume is greater. This factor (in our example, 8) can be decomposed into two components: the factor by which velocity is greater, $V_B^{2/3}$, and the factor by which the number of shares traded per bet is greater, $V_B^{1/3}$. This decomposition is apparent in Exhibit 1. Betting volume for the high-velocity stock is eight times greater than for the low-velocity stock ($16{,}000{,}000 \div 2{,}000{,}000 = 8$). If we raise 8 to the 2/3 power, we get the factor by which velocity is greater ($8^{2/3} = 4$ and $400 \div 100 = 4$). And if we raise 8 to the 1/3 power, we get the factor by which quantity is greater ($8^{1/3} = 2$ and $40{,}000 \div 20{,}000 = 2$). To the extent betting activity is greater, velocity is greater by a factor that is always twice as high as the factor by which quantity is greater, as long as both stocks have the same betting volatility.

Visualization of Risk Transfer

It may be helpful to present the invariance of risk transfer visually. Empirical analysis shows that the distribution of unsigned risk transfer is close to lognormal; it is symmetric for purchases and sales. We therefore map the values along the horizontal axis to probability densities using the following function, where σ_I equals the standard deviation of the natural logarithm of the absolute value of quantity. In Equation 17, I_B is an absolute value.

$$D(I_B) = \frac{1}{I_B \sigma_I \sqrt{2\pi}} e^{-\frac{(\ln I_B - \mu_I)^2}{2\sigma_I^2}} \tag{17}$$

The constants, log-mean μ_I and log-variance σ_I^2, are calibrated from the data and are equal to −5.71 and 2.50,

EXHIBIT 1
Invariance of Risk Transfer

		Mean Bet		Median Bet		+ 2 σ Bet	
	Invariance Parameters	**Low γ**	**High γ**	**Low γ**	**High γ**	**Low γ**	**High γ**
γ	Velocity	100	400	100	400	100	400
$1/\gamma$	Business time/calendar time	1.00%	0.25%	1.00%	0.25%	1.00%	0.25%
P	Price	\$40	\$40	\$40	\$40	\$40	\$40
Q	Quantity of shares per bet	20,000	40,000	5,620	11,240	135,300	270,600
$P \cdot Q$	Bet size	\$800,000	\$1,600,000	\$224,800	\$449,600	\$5,412,000	\$10,824,000
ς	Volume multiplier	3.0	3.0	3.0	3.0	3.0	3.0
ψ	Volatility multiplier	0.64	0.64	0.64	0.64	0.64	0.64
σ_R	Return volatility (daily)	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%
V_B	Betting volume	2,000,000	16,000,000	2,000,000	16,000,000	2,000,000	16,000,000
V_T	Trading volume	3,000,000	24,000,000	3,000,000	24,000,000	3,000,000	24,000,000
σ_B	Betting volatility (daily)	1.28%	1.28%	1.28%	1.28%	1.28%	1.28%
σ_T	Betting volatility (business time)	0.128%	0.064%	0.128%	0.064%	0.128%	0.064%
W_T	Trading activity	\$2,400,000	\$19,200,000	\$2,400,000	\$19,200,000	\$2,400,000	\$19,200,000
W_B	Betting activity	\$1,024,000	\$8,192,000	\$1,024,000	\$8,192,000	\$1,024,000	\$8,192,000
$\boldsymbol{I_B}$	**Risk transfer per bet**	**\$1,024**	**\$1,024**	**\$288**	**\$288**	**\$6,927**	**\$6,927**

respectively. For sizes of bet Q, the distributions are still lognormal but with different means.

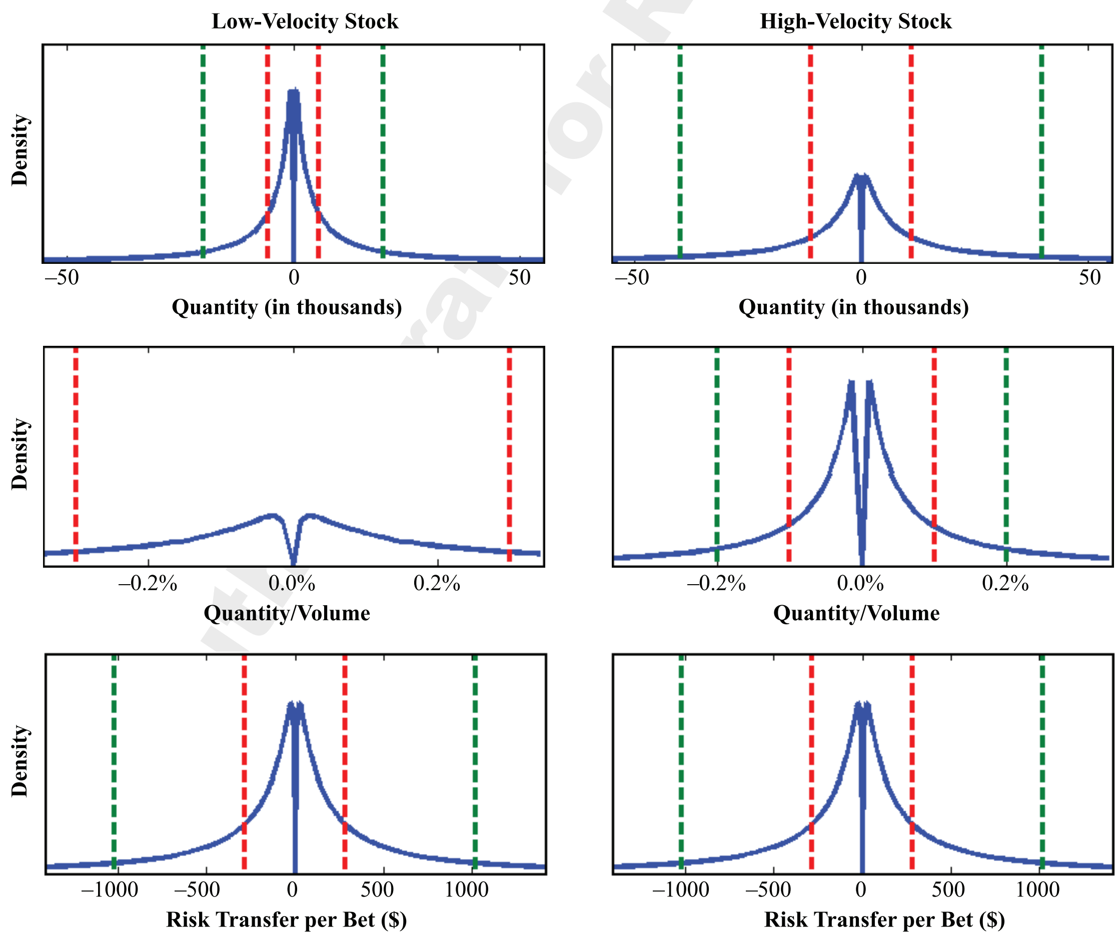
The top panel of Exhibit 2 shows the quantity of shares traded per bet Q for the low-velocity stock and the high-velocity stock. The left side of each graph shows the distribution for sales, and the right side shows the distribution for purchases. The green vertical lines represent the quantity of shares traded for the mean bets, whereas the red vertical lines pertain to the median bets. Notice that the locations of these lines correspond to the values in Exhibit 1 for the quantity of shares traded. There is one other noteworthy feature in the top panel of Exhibit 2. Faster markets have larger bets. The density of the distribution of quantity for the high-velocity stock is further from zero than for the low-velocity stock by a factor of 2, which we earlier showed would be the case.

The middle panel of Exhibit 2 shows the quantity of shares traded per bet as a fraction of betting volume, $\frac{Q}{V_B}$. Now notice that when quantity is divided by betting volume, the density of the distribution of the low-velocity stock is further from zero than for the high-velocity stock by a factor of 4, as we also showed earlier would be the case. Even though unsigned bet size in shares is smaller for the low-velocity stock than for the high-velocity stock, the bigger volume in the denominator for the latter dominates the ratio and reverses the ranking.

Finally, the bottom panel shows that the distribution of risk transferred by betting

$$I_B = P \cdot Q \cdot \sigma_B \cdot \gamma^{-\frac{1}{2}}$$

EXHIBIT 2
Visualization of the Invariance of Risk Transfer



is identical for both the low- and high-velocity stocks, despite the fact that the quantity of shares traded per bet as an absolute value and relative to betting volume differs for each stock by a factor of 2 and 1/4, respectively. This invariance arises because the dollar value of risk is simultaneously rescaled by the square root of velocity.

Transaction Costs

Exhibit 3 offers a numerical illustration of the invariance of transaction costs. We assume the same mean transfer of risk per bet and stock-specific amounts of risk transfer as we did in the risk transfer example.

According to the invariance hypothesis, the dollar-cost function is the same across low- and high-velocity stocks. Suppose it equals \$720 for the mean bet, \$120 for the median bet, and \$11,600 for a bet two standard deviations above the mean. This assumes the square-root price impact function with spread, presented later in Equation 18. We also assume that the average cost for transferring risk $\overline{C_B}$ is \$1,680. These values have been calibrated from the data.[6] Recall that the percentage cost of risk transfer $C(Q)$ is a function of Q. The average cost of risk transfer $\overline{C_B}$ does not equal the cost of risk transfer for the mean bet owing to Jensen's inequality,

$$\left(\overline{C_B(I_B)} = \$1,680 \neq \$720 = C_B(\overline{I_B})\right).$$

We also assume the same mean bet size and stock-specific bet sizes as we did in the risk transfer example. The mean risk transfer and the mean bet size obviously are equal for mean bets, median bets, and standard deviation bets.

In Exhibit 3, we compute the percentage cost of risk transfer $C(Q)$ by dividing the dollar cost by the size of the bet. For the mean bet, it is 0.090% (\$720 ÷ \$800,000) for the low-velocity stock and 0.045% (\$120 ÷ \$224,800) for the high-velocity stock.

We next define asset-specific illiquidity as the average cost of transferring risk per mean bet. It is equal to 0.210% (\$1,680 ÷ \$800,000) for the low-velocity stock and 0.105% (\$1,680 ÷ \$1,600,000) for the high-velocity stock.

If we divide percentage cost by illiquidity $\frac{1}{L}$, we get the invariant average cost function $f(I_B)$, which equals 0.43 for both the low-velocity stock (0.090% ÷ 0.210%) and the high-velocity stock (0.045% ÷ 0.105%). It follows, therefore, that we can describe the percentage cost function as the product of asset-specific illiquidity and the invariant average cost function. These calculations also apply to the median and two-standard-deviation bet. Notice that the average costs are the same for both median bets (0.25) and both two-standard-deviation bets (1.01). Exhibit 4 summarizes these calculations.

Visualization of Transaction Costs

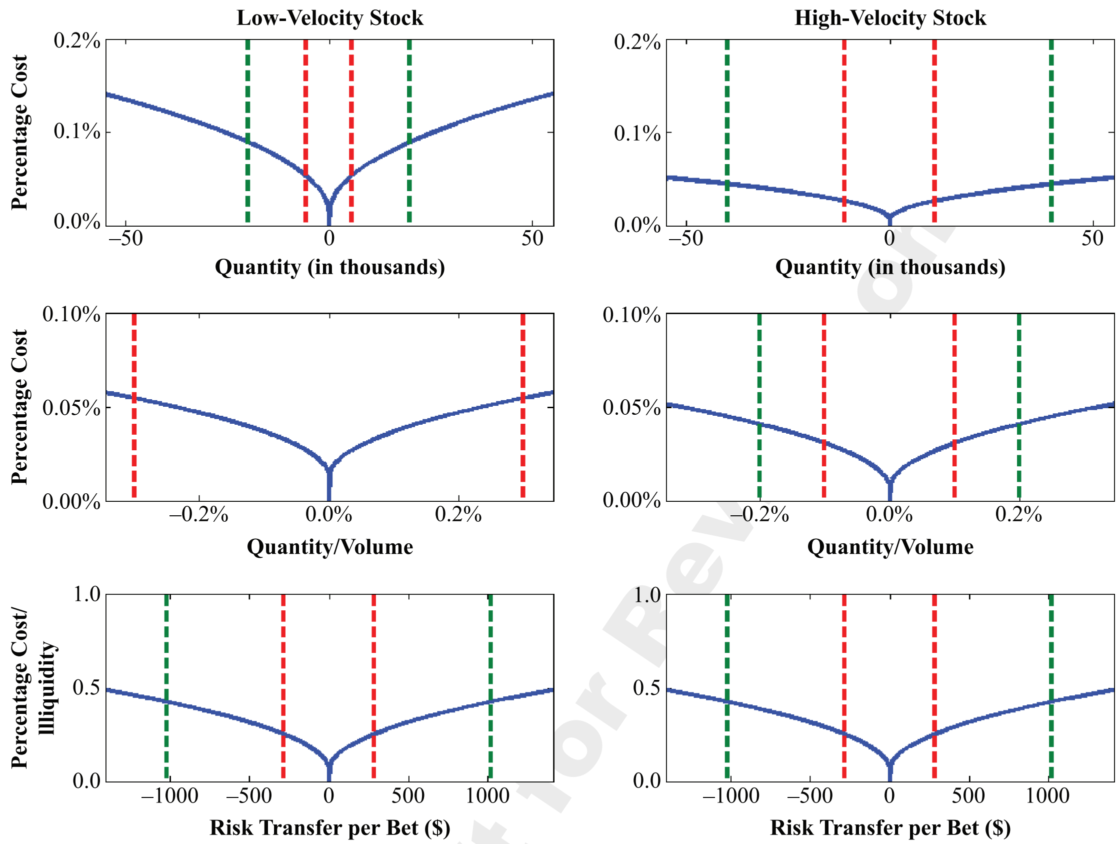
Empirical analysis implies the following transaction cost function:

$$C(Q) = \frac{\sigma_B}{0.02}\left(\frac{1}{2}\bar{k}\cdot\sqrt[3]{\frac{W_{B,\text{Benchmark Stock}}}{W_B}} + \frac{1}{2}\bar{\lambda}\cdot\sqrt[2]{\frac{Q}{0.01V_B}}\right). \quad (18)$$

E X H I B I T 3
Invariance of Transaction Costs

	Invariance Parameters	Mean Bet Low γ	Mean Bet High γ	Median Bet Low γ	Median Bet High γ	+ 2 σ Bet Low γ	+ 2 σ Bet High γ
$\boldsymbol{I_B}$	**Stock-specific risk transfer per bet**	**\$1,024**	**\$1,024**	**\$288**	**\$288**	**\$6,927**	**\$6,927**
$\boldsymbol{C_B(I_B)}$	**Invariant cost of risk transfer across stocks**	**\$720**	**\$720**	**\$120**	**\$120**	**\$11,500**	**\$11,500**
$\boldsymbol{E\|I_B\|}$	**Risk transfer per mean bet**	**\$1,024**	**\$1,024**	**\$1,024**	**\$1,024**	**\$1,024**	**\$1,024**
$E\|PQ\|$	Mean bet size	\$800,000	\$1,600,000	\$800,000	\$1,600,000	\$800,000	\$1,600,000
$\boldsymbol{C_B}$	**Mean cost of risk transfer**	**\$1,680**	**\$1,680**	**\$1,680**	**\$1,680**	**\$1,680**	**\$1,680**
$P \cdot Q$	Bet size	\$800,000	\$1,600,000	\$224,800	\$449,600	\$5,412,000	\$10,824,000
$C(Q)$	Percentage cost of risk transfer	0.090%	0.045%	0.053%	0.027%	0.212%	0.106%
$1/L$	Stock-specific illiquidity	0.210%	0.105%	0.210%	0.105%	0.210%	0.105%
$\boldsymbol{f(I_B)}$	**Invariant average cost function**	**0.43**	**0.43**	**0.25**	**0.25**	**1.01**	**1.01**

EXHIBIT 4
Visualization of the Invariance of Transaction Costs



The constants, $\frac{1}{2}\bar{k}$ and $\frac{1}{2}\bar{\lambda}$, are calibrated from the data and are equal to 2.08×10^{-4} and 12.08×10^{-4}, respectively. For the benchmark stock with $W_{B,\, Benchmark\ Stock}$ = \$40 $\cdot$ 10^{-6} $\cdot$ 0.002, the half-spread equals 2.08 basis points, and the price impact of trading 1% of volume equals 12.08 basis points. Invariance does not place any restrictions on the shape of the transaction cost function, but the square-root specification matches the data better than a linear one for a wide range of bet sizes.[7]

In the top panel of Exhibit 4, X on the horizontal axis is mapped onto Q, the quantity of shares per bet, and Y is percentage cost $C(Q)$. In the middle panel, X is mapped onto to $\frac{Q}{V_B}$, the quantity of shares per bet scaled by betting volume, and Y is percentage cost $C\left(\frac{Q}{V_B}\right)$. In the bottom panel, X is again equal to risk transfer per bet, I_B; percentage cost is then divided by $\frac{1}{L}$.

The top panel of Exhibit 4 shows the percentage cost of risk transfer $C(Q)$ for a given quantity of shares traded per bet. Notice that these costs match the mean and median costs shown in Exhibit 3, and that the costs for low-velocity bets are substantially greater than for high-velocity bets, given the same quantity of shares traded. Bets in a high-velocity stock have lower transaction costs because higher velocity is associated with more efficient pricing.

The middle panel of Exhibit 4 shows the percentage cost of risk transfer $C(Q)$ for a given quantity traded as a fraction of betting volume, $C\left(\frac{Q}{V_B}\right)$. Notice that the percentage costs as a function of Q, both in absolute terms (top panel) and as a fraction of betting volume (middle panel), differ for low-velocity and high-velocity stocks; in the middle panel, the functions have slightly different intercepts.

Now, observe the bottom panel. It reveals that when percentage costs are adjusted in two ways—(1) the argument Q is transformed into I_B, and (2) percentage costs $C(Q)$ are scaled by the stock-specific illiquidity measure $\frac{1}{L}$ to be transformed into the average cost function $f(I_B)$, as in Equation 16—then the adjusted cost of risk transfer is the same for low- and high-velocity stocks. In these graphs, the horizontal axis is mapped onto I_B, and the vertical axis is mapped onto

$$C(Q)\Big/\left(\frac{1}{L}\right).$$

The plots for dollar cost functions $C_B(I_B)$ will also be the same for low- and high-velocity stocks.

These exhibits illustrate that many variables vary significantly across low- and high-velocity stocks—but by applying appropriate scaling, we uncover invariance relationships.

SUMMARY

We have presented a comprehensive model of market microstructure invariance. This model follows logically from three simple transformations, together with two hypotheses. First, we transformed trades to bets, calendar time to business time, and return volatility to dollar volatility. Next, we hypothesized that the amount of risk transferred per bet is the same for low-velocity and high-velocity stocks because, holding other attributes constant, bets with higher velocity involve a greater number of shares traded. We also hypothesized that the dollar cost of executing low- and high-frequency bets is the same when measured as a function of the dollar amount of risk transferred by the bets. In addition, the percentage cost of transferring risk is also the same when adjusted to account for stock-specific illiquidity.

We presented a detailed system of equations to represent this model, and we illustrated these mathematical relationships both numerically and visually. We refer readers to Kyle and Obizhaeva [2016a, 2016b] for additional insight into market microstructure invariance and, in particular, for a comprehensive compilation of significant empirical evidence to support these invariance hypotheses.

APPENDIX

Mathematics of Exponents

In order to understand the derivation of Equations 9, 10, 11, 12, and 15, it may be useful to review the mathematics of exponents:

$$x^{-n} = \frac{1}{x^n} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad (xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Derivation of Equations

Betting activity, as defined in Equation 9, is derived as follows:

$$W_B = \sigma_B \cdot PV_B \qquad \text{from Equation 6}$$

$$W_B = \sigma_B \cdot P \cdot \gamma \cdot E|Q| \qquad \text{from Equation 1}$$

$$E|Q| = \gamma^{\frac{1}{2}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot E|I_B| \qquad \text{from Equation 8}$$

$$W_B = \sigma_B \cdot P \cdot \gamma \cdot \gamma^{\frac{1}{2}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot E|I_B| \qquad \text{rearranging}$$

$$W_B = \gamma \cdot \gamma^{\frac{1}{2}} \cdot E|I_B| \qquad \text{cancelling out } \sigma_B \text{ and } P$$

$$W_B = \gamma^{\frac{3}{2}} \cdot E|I_B| \qquad \text{consolidating}$$

Velocity, as defined in Equation 10, is derived as follows:

$$W_B = \gamma^{\frac{3}{2}} \cdot E|I_B| \qquad \text{from Equation 9}$$

$$\gamma^{\frac{3}{2}} = \frac{W_B}{E|I_B|} \qquad \text{rearranging}$$

$$\left(\gamma^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(\frac{W_B}{E|I_B|}\right)^{\frac{2}{3}} \qquad \text{raising both sides to the 2/3 power}$$

$$\gamma = W_B^{\frac{2}{3}} \cdot (E|I_B|)^{\frac{-2}{3}} \qquad \text{rearranging}$$

The average number of shares traded per bet, as defined in Equation 11, is derived as follows:

$$E|Q| = \gamma^{\frac{1}{2}} \cdot E|I_B| \cdot P^{-1} \cdot \sigma_B^{-1} \qquad \text{from Equation 8}$$

$$E|Q| = \left(W_B^{\frac{2}{3}} \cdot E|I_B|^{\frac{-2}{3}} \right)^{\frac{1}{2}} \cdot E|I_B| \cdot P^{-1} \cdot \sigma_B^{-1} \quad \text{from Equation 10}$$

$$E|Q| = W_B^{\frac{1}{3}} \cdot \frac{1}{P\sigma_B} \cdot \left(E|I_B|\right)^{\frac{2}{3}} \quad \text{rearranging.}$$

The distribution of the number of shares traded as a fraction of betting volume, as defined in Equation 12, is derived as follows:

$$Q = \gamma^{\frac{1}{2}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot I_B \quad \text{from Equation 8}$$

$$Q = \left(W_B^{\frac{2}{3}} \cdot E|I_B|^{\frac{-2}{3}} \right)^{\frac{1}{2}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot I_B \quad \text{from Equation 10}$$

$$Q = W_B^{\frac{1}{3}} \cdot \left(E|I_B|\right)^{\frac{-1}{3}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot I_B \quad \text{rearranging}$$

$$\frac{Q}{V_B} = W_B^{\frac{1}{3}} \cdot \left(E|I_B|\right)^{\frac{-1}{3}} \cdot P^{-1} \cdot \sigma_B^{-1} \cdot V_B^{-1} \cdot I_B \quad \text{dividing by } V_B$$

$$\frac{Q}{V_B} = W_B^{\frac{1}{3}} \cdot \left(E|I_B|\right)^{\frac{-1}{3}} \cdot W_B^{-1} \cdot I_B \quad \text{from Equation 6}$$

$$\frac{Q}{V_B} = W_B^{\frac{-2}{3}} \cdot \left(E|I_B|\right)^{\frac{-1}{3}} \cdot I_B \quad \text{rearranging.}$$

The stock-specific illiquidity measure in Equation 15 is derived as follows:

$$\frac{1}{L} = P^{-1} \cdot \left(E|Q|\right)^{-1} \cdot C_B \quad \text{from Equation 14}$$

$$E|Q| = W_B^{\frac{1}{3}} \cdot \frac{1}{P\sigma_B} \cdot \left(E|I_B|\right)^{\frac{2}{3}} \quad \text{from Equation 11}$$

$$\frac{1}{L} = P^{-1} \cdot \left(W_B^{\frac{1}{3}} \cdot \frac{1}{P\sigma_B} \cdot \left(E|I_B|\right)^{\frac{2}{3}} \right)^{-1} \cdot C_B \quad \text{rearranging}$$

$$\frac{1}{L} = \sigma_B \cdot W_B^{\frac{-1}{3}} \cdot \left(E|I_B|\right)^{\frac{-2}{3}} \cdot C_B \quad \text{rearranging.}$$

The decomposition of the percentage cost function in Equation 16 is derived as follows:

$$C(Q) = \frac{C_B(I_B)}{PQ} \quad \text{from Equation 13}$$

$$C(Q) = \frac{C_B}{E|PQ|} \cdot \frac{C_B(I_B)}{C_B} \cdot \frac{E|PQ|}{PQ} \quad \text{rearranging}$$

$$C(Q) = \frac{C_B}{E|PQ|} \cdot \frac{C_B(I_B)}{C_B} \cdot \frac{E|I_B|}{I_B} \quad \text{from Equation 4}$$

$$C(Q) = \frac{1}{L} \cdot \frac{C_B(I_B)}{C_B} \cdot \frac{E|I_B|}{I_B} \quad \text{from Equation 14.}$$

ENDNOTES

We thank Cel Kulasekaran and Olya Obizhaeva for computational assistance and helpful comments.

[1]The mathematical derivation presented in this section is not essential for understanding the key insights of market microstructure invariance.

[2]See Kyle and Obizhaeva [2016b] for more information.

[3]For more detail about calibration, see Kyle and Obizhaeva [2016a].

[4]For more detail about calibration, see Kyle and Obizhaeva [2016a].

[5]See Kyle and Obizhaeva [2016a] for more detail.

[6]For more detail about calibration, see Kyle and Obizhaeva [2016a].

[7]The linear specification has the form

$$C(Q) = \frac{\sigma_B}{0.02}\left(\frac{1}{2}\bar{k} \cdot \sqrt[3]{\frac{W_{B,Benchmark\ Stock}}{W_B}} + \frac{1}{2}\bar{\lambda} \cdot \sqrt[3]{\frac{W_B}{W_{B,Benchmark\ Stock}}}\ \frac{Q}{0.01V_B} \right).$$

The constants, $\frac{1}{2}\bar{k}$ and $\frac{1}{2}\bar{\lambda}$, are calibrated from the data and are equal to 8.21×10^{-4} and 2.50×10^{-4}, respectively. For the benchmark stock with $W_{B,Benchmark\ Stock} = \$40 \cdot 10^6 \cdot 0.02$, the half-spread equals 8.21 basis points, and the price impact of trading 1% of volume equals 2.50 basis points. See Kyle and Obizhaeva [2016a] for more detail.

REFERENCES

Kyle, A.S., and A.A. Obizhaeva. "Market Microstructure Invariance: Empirical Hypotheses." *Econometrica*, Vol. 84, No. 4 (2016a), pp. 1345-1404.

——. "Market Microstructure Invariance: A Dynamic Equilibrium Model." February 8, 2016b, SSRN 2749531.