Beliefs Aggregation and Return Predictability

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We study return predictability using a dynamic model of speculative trading among relatively overconfident competitive traders who agree to disagree about the precision of their private information. The return process depends on both parameter values used by traders and empirically correct parameter values. Although traders apply Bayes Law consistently, equilibrium returns are predictable. Parameters are calibrated to generate empirically realistic patterns of short-run momentum and long-run mean-reversion. Consistent with our model's prediction, our empirical tests show that time-series return momentum is more pronounced for stocks with higher trading volume.

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While it is well known empirically that individual stock returns exhibit momentum (positive autocorrelation) over short time periods and mean reversion (negative autocorrelation) related to a value-growth anomaly over long time periods, researchers have found it difficult to explain theoretically why these time series patterns occur. We describe a competitive dynamic model in which relatively overconfident traders disagree about the precision of signals. Greater disagreement leads to more momentum, trading volume, and liquidity. We calibrate model parameters to fit the short-term momentum and long-term mean reversion observed in the data. We also confirm empirically the prediction that stocks with higher trading volume exhibit more momentum.

Our model uses a structure similar to the smooth trading model of Kyle, Obizhaeva and Wang (2017) but makes several significant modifications. First, instead of focussing on how imperfect competition affects trading costs and quantities traded, we focus on how perfect competition affects returns, like Kyle and Lin (2002). This makes it possible to derive most results analytically and to show that momentum arises naturally even in a setting of perfect competition. Second, instead of focussing on equilibrium properties from the perspective of traders regardless of whether their beliefs are empirically correct, we emphasize the importance of empirically correct beliefs, paying particular attention to the case when traders are correct on average. This makes it possible to show how return predictability depends both on traders' beliefs in the model and empirically correct beliefs, along the lines of Xiong and Yan (2010). Third, instead of setting the model in continuous time, the model is set in discrete time. This makes it possible to show that momentum is stronger when trading opportunities are more frequent. Fourth, instead of assuming that there is one unobserved component of dividend growth about which traders have private information, the model assumes that dividend growth has two components, an observed long-term component and an unobserved short-term component about which traders have private signals. Our model generates a short-term momentum anomaly due to speculative trading on disagreement about a short-term growth rate. It also generates a value-growth anomaly by assuming that all traders equally overestimate the persistence of a long-term growth rate that they all observe. This makes it possible to match empirically realistic patterns of short-run momentum and long-run reversal.

Like asset managers in real markets, traders in the model act as if they collect public and private raw information into databases, engage in research to process this information into signals, calculate expected returns or alphas from these signals, and construct optimal inventories by inputting alphas into risk models. The traders are "relatively overconfident" in that each symmetrically assigns a higher value to the precision to his own private signal than the precision of other traders' signals. Since the values traders assign to all economically relevant parameters are common knowledge, traders agree to disagree about the informativeness of their respective signals (Aumann (1976)). We show that expected returns are predictable, even when the beliefs of traders are "correct on average" in the sense that traders have correct beliefs about the average precision of all signals, including their own. This theoretical result contradicts the empirical rational expectations intuition that prices will aggregate fundamental information correctly when traders are correct on average, even when individual traders make mistakes (Muth (1961)).

Beliefs Aggregation. Our model highlights mechanisms generating momentum and return predictability in an infinite horizon, stationary model in which traders disagree about the precision of private signals. Return predictability first of all results from current prices being dampened. Price dampening arises from two conceptually different effects that we call static and dynamic beliefs aggregation.

First, *static beliefs aggregation* dampens current prices due to the way in which traders' average expectations are dampened with relatively overconfident beliefs. Traders place weights on signals proportional to the square root of the signal's perceived precision. In equilibrium, this makes the market weight on each signal proportional to the average of the square roots of the precisions across traders. Under the assumption that traders' beliefs are empirically correct on average, the empirically correct signal weight is the square root of the average of the precisions across traders, not the average of the square roots. Since Jensen's inequality implies that the average of the square roots is less than the square root of the averages, heterogeneous weights dampen the price, making it underreact to the total amount of private information available in the market and revealed in prices due to symmetry.

Second, *dynamic beliefs aggregation* dampens prices due to incentives for short-term trading resulting in weights on traders' valuations summing to less than one. Traders not only agree to disagree about the value of the asset in the present, but they also agree to disagree about how their valuations will change in the future. Each trader believes that others traders' valuations will mean revert faster than implied by the trader's own beliefs, even when the other traders' valuations coincidentally happen to be correct. To exploit this perceived mean reversion in other traders' valuations, each trader engages in short-term trading, betting against other traders' valuations and exploiting how they believe these valuations will change in the near future. Since this short-term trading sometimes makes traders take positions opposite to what their own long-term valuations imply, this mechanism reflects the logic of a *Keynesian beauty contest*. Even if a trader thinks it is profitable to buy the asset based on its long run fundamental value, he may instead sell the asset for short-term speculative motives because he thinks that other traders place too much weight on their signals and will revise their valuations downward in the short run.

Short-term trading dampens prices in the sense that prices are not equal to the average of traders' buy-and-hold valuations—even though there is no noise trading—because the weights in the average of valuations sum up to less than one. In contrast, the weights sum up to exactly one in an analogous one-period model. The dynamic dampening effect also goes away in dynamic models when traders can only buy and hold. While dynamic dampening becomes more substantial when traders can trade more frequently, static dampening does not depend on traders' trading frequency and may arise in a one-period model.

While traders' beliefs affect current prices, the properties of return dynamics, such as autocorrelations at different horizons, also depend on the empirically correct model specification for dividends, value-relevant non-dividend information, and private information, which ultimately govern the actual dynamics of prices and returns. We assume that traders use models with correct structure but with possibly incorrect parameters. This makes returns a function of both the parameters traders use and the correct model parameters. Our analysis focusses on two mistakes that traders make. First, we assume that traders are relatively overconfident about their own signals in comparison with the signals of other traders but are correct on average about the total informativeness of all signals combined. We show that this generates momentum. Second, we assume that traders overestimate the persistence of dividend growth so that prices overreact to the long-term growth rate and exhibit mean-reversion, generating a time-series value effect. More generally, while price dampening due to relative overconfidence creates a tendency for momentum in returns, overall return dynamics may in principle be influenced by many factors, which lead to a complicated autocorrelation structure.

Theoretical Literature. Our paper is related to the literature on beliefs and information aggregation. Allen, Morris and Shin (2006) show that when traders have a common prior and therefore agree about the precision of signals, iterating expectations taken over different information sets leads to momentum. In a noisy rational expectations version of their model, asymmetric information and price drift are associated with excess volatility and mean reversion, not momentum (Banerjee, Kaniel and Kremer (2009)). When noise vanishes, the weights on traders' valuations sum to one and asymmetric information disappears because traders can infer the average signal from prices; there is no dampening and no momentum. Since our model does not have noise trading, traders also infer the average expectation from prices. Unlike the model of Allen, Morris and Shin (2006), dampening arises in our model because expectations are averaged across traders with different beliefs and the same information set, not because expectations are averaged across traders with the same beliefs and different information sets.

In a related paper, Banerjee, Kaniel and Kremer (2009) investigate a fully symmetric, CARA-normal model with two rounds of trading in which no public information is available and no dividends are paid out when trading takes place, traders have empirically correct beliefs about the precision of their own information, and traders believe other traders' information to be less precise than their own. They point out that for momentum to occur, it is necessary for traders to disagree about future valuations of fundamentals. Their analytical derivations do not make clear to what extent momentum—when it occurs—is related to dampening from static and dynamic beliefs aggregation, incorrect beliefs about others' signal precision, and the absence of new public information when trade occurs. Our dynamic steady-state model has no noise trading, has disagreement about the distribution of private signals, has public and private information arriving every period, and allows traders to believe their own and others signals are either more or less precise than is empirically the case. We characterize precisely when momentum occurs and show how it depends on these different assumptions.

Daniel, Hirshleifer and Subrahmanyam (1998) obtain excess volatility and mean reversion in a representative agent model when the representative agent is "absolutely overconfident," believing information to be more precise than is empirically the case. In contrast, we obtain momentum when traders are relatively overconfident but correct on average. While absolute overconfidence tends to generate excess volatility and mean reversion, relative overconfidence tends to generate momentum.

The assumptions of zero-net-supply and constant absolute risk aversion approximate markets for individual stocks, where risks are idiosyncratic and wealth effects are not significant. By contrast, the interaction of beliefs aggregation with wealth effects, without private information, are the focus of Detemple and Murthy (1994), Basak (2005), Jouini and Napp (2007), Dumas, Kurshev and Uppal (2009), Xiong and Yan (2010), Cujean and Hasler (2014), Atmaz and Basak (2015), and Ottaviani and Sorensen (2015). Andrei and Cujean (2017) focus on word-of-mouth communication instead of beliefs aggregation as a mechanism that generates return predictability. Conceptually, our approach is most similar to the approach of Campbell and Kyle (1993), who use noise trading to generate excess volatility and mean reversion instead of relative overconfidence to generate momentum.

Return predictability in our paper is not related to changes in the aggregate amount of money chasing the return on the risky asset, as in Gruber (1996), Lou (2012), and Vayanos and Woolley (2013). Due to market clearing the aggregate flow of money into the market for risky assets is zero, even though individual traders indeed find profitable investment opportunities and chase returns.

It is fashionable to attribute predictability in asset returns to irrational behavior motivated by psychology. This presumes that rational behavior instead would lead to no return predictability. Simon (1957) proposes the concept of bounded rationality for studying the irrationality of human choices resulting from various institutional constraints such as the psychological costs of acquiring information, cognitive limitations of human minds, or the finite amount of time humans have to make a decision. Hong and Stein (1999), Barberis and Shleifer (2003), and Greenwood and Shleifer (2014) assume that traders follow simple trading rules and do not extract information from prices. When return anomalies are motivated by behavioral biases, Fama (1998) suggests that a Pandora's box is opened, undermining modeling parsimony by enabling one plethora of behavioral biases to explain another plethora of anomalies.

To motivate trade, we relax the common prior assumption in a minimal way. Traders are willing to trade because they believe their private signals are more precise than their competitors believe them to be. Except for this relative overconfidence, traders are otherwise completely rational. They apply Bayes Law consistently and optimize correctly. No additional behavioral assumptions or modeling ingredients, like noise trading, are needed to generate trade. Our paper follows Morris (1995), who eloquently argues for "dropping the common prior assumption from otherwise rational behavior" models as an important and largely overlooked modeling approach, since even rational agents may have heterogeneous beliefs.

Empirical Literature. Our model provides a formal economic underpinning for the extensive empirical literature studying the predictability of returns at different horizons using past price, book value, and measures of cash flow (dividends). The expected returns are linear functions of state variables including the current levels of prices, dividends, and long-term dividend growth rates as well as exponentially weighted averages of these variables in the past. The decay rates of past prices and dividends are proportional to the informativeness of prices, measured by the total precision of information. Our model thus places specific testable non-linear micro-founded economic restrictions on VAR models of expected returns such as Goyal and Welch (2003), Ang and Bekaert (2007), Cochrane (2008), Van Binsbergen and Koijen (2010), and Rytchkov (2012). These restrictions are sufficiently flexible to be consistent with the rich patterns of short-term momentum and long-term mean-reversion. Our analysis provides some guidance for empirical research on return predictability in markets with heterogenous beliefs, such as Greenwood and Shleifer (2014) and Buraschi, Piatti and Whelan (2016).

In a simple calibration exercise, we show that realistic model parameters can be chosen to match closely the observed levels of positive returns autocorrelation over short periods of one to two years and negative autocorrelation over longer periods.

Our theoretical predictions are consistent with some empirical findings on properties of momentum patterns. For example, we show empirically that momentum patterns tend to be more pronounced for stocks with more trading. Also, Lee and Swaminathan (2000) and Cremers and Pareek (2014) document that momentum is stronger for stocks with higher trading volume and short-term trading. Moskowitz, Ooi and Pedersen (2012) find that more liquid futures contracts tend to exhibit more momentum. Zhang (2006) and Verardo (2009) show that momentum returns are larger for stocks with higher analysts' disagreement. Similar properties characterize momentum patterns in our model, because price dampening tends to be more substantial when there is more disagreement.

This paper is structured as follows. Section 1 presents a competitive model with discrete trading. Section 2 analyzes holding-period returns as functions of *both* an empirically correct specification of fundamentals and information *and* possibly empirically incorrect beliefs of traders. Section 3 calibrates the model parameters and conducts some empirical analysis. Section 4 concludes. All proofs are in the Appendix.

1. The Model

We first describe a competitive model in which information arrives continuously but trading takes place at discrete intervals. The price aggregates traders' heterogeneous beliefs about how information should be correctly processed. Given their individual beliefs, traders behave in a rational manner. They collect public and private information, construct signals from the information, apply Bayes Law correctly to predict returns, and calculate optimal holdings. Trader are collectively irrational in that each trader is relatively overconfident, believing that the precision of his own private information flow is greater than other traders believe it to be.

1.1. Model Assumptions

Both fundamentals and information evolve continuously over the time interval $t \in (-\infty, \infty)$. Trading takes place at discrete dates $t = kh$, where k indexes time periods $k = \dots, -2, -1, 0, 1, 2, \dots$, and $h > 0$ is the time interval between each round of trading. Varying h makes it possible to examine how the frequency of trading affects the equilibrium when the continuous flow of information does not change. At $t = kh$, N risk-averse perfect competitors trade a risky asset against a risk-free asset at price P_k. The risky security is in zero net supply, and the risk-free asset earns constant risk-free rate $r > 0$.

The risky asset pays dividends at continuous rate $D(t)$. Dividends follow a stochastic process with stochastic long-term growth rate $G_L(t)$ and short-term growth rate $G_S^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$:

$$dD(t) := -\alpha_D \, D(t) \, dt + G_L(t) \, dt + G_S^*(t) \, dt + \sigma_D \, dB_D(t). \tag{1}$$

The long-term growth rate $G_L(t)$ follows an AR-1 process with mean-reversion α_L and volatility σ_L:

$$dG_L(t) := -\alpha_L \, G_L(t) \, dt + \sigma_L \, dB_L(t). \tag{2}$$

The short-term growth rate $G_S^*(t)$ follows an AR-1 process with mean-reversion α_S and volatility σ_S:

$$dG_S^*(t) := -\alpha_S \, G_S^*(t) \, dt + \sigma_S \, dB_S(t). \tag{3}$$

The dividend $D(t)$ and the long-term growth rate $G_L(t)$ are publicly observable. The short-term growth rate $G_S^*(t)$, marked with a star superscript, is not observed by traders. If the dividend $D(t)$, the long-term growth rate $G_L(t)$, and the short-term growth rate $G_S^*(t)$

were observable, then the price of the asset would equal its fundamental value given by the generalization of the Gordon growth formula

$$F(t) = \frac{D(t)}{r+\alpha_D} + \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)} + \frac{G_S^*(t)}{(r+\alpha_D)(r+\alpha_S)}. \tag{4}$$

For empirical interpretation, the variable $D(t)$ corresponds to dividends or "cash flow," and $D(t)/(r+\alpha_D)$ corresponds to "book value." Since the model is arithmetic and not geometric, the difference between price and the book value is analogous to a market-to-book ratio. If price is greater than the book value, the firm is a growth stock; otherwise, the firm is a value stock.

Each trader observes public and private signals about the short-term growth rate $G_S^*(t)$, then constructs an estimate of the fundamental value $F(t)$ by replacing $G_S^*(t)$ in equation (4) with its expectation. As shown below, the equilibrium price looks like equation (4) with $G_S^*(t)$ replaced by a weighted sum of traders' estimates of $G_S^*(t)$ with weights summing to less than one.

Each trader n observes a continuous stream of private information $I_n(t)$ about the scaled, unobservable, short-term growth rate $G_S^*(t)$:

$$dI_n(t) := \tau_n^{1/2} \, \frac{G_S^*(t)}{\sigma_S \, \Omega^{1/2}} \, dt + dB_n(t). \tag{5}$$

The parameter Ω is a scaling constant discussed below (equation (8)); the parameter τ_n measures the informativeness of $dI_n(t)$. Each increment $dI_n(t)$ in equation (5) is a noisy observation of the unobserved growth rate $G_S^*(t)$ since its drift is proportional to $G_S^*(t)$. No noise trading implies that traders infer the average estimate from the price.

Each trader is certain that his own private information $I_n(t)$ has high precision $\tau_n = \tau_H$, and the other traders' private information has low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$. Since this disagreement is common knowledge, relatively overconfident traders agree to disagree about the precision of their signals.[1]

[1] Consider rescaling the private information (5) as a scaled growth rate plus noise, $dI_n(t) = G_S^*(t)/(\sigma_S \, \Omega^{1/2}) \, dt + \tau_n^{-1/2} dB_n(t)$, $n = 0, 1, \ldots, N$, so that trader n observes $\tau_n^{-1/2} \, dI_n(t)$ rather than $dI_n(t)$. This changes the equilibrium because traders disagree about whether to use factors $\tau_H^{1/2}$ or $\tau_L^{1/2}$ to convert one scaling into the other. We solved the equilibrium and found that the dampening effect and returns predictability, discussed below, disappear under this different scaling. Since a trader can estimate the diffusion variance with high accuracy by observing $dI_n(t)$ over short time intervals in continuous time, equation (5) has the appealing scaling that traders infer the correct diffusion variance while rescaling has the unappealing feature that the observed diffusion variance will contradict some traders' beliefs about it.

Each trader also makes inferences about the growth rate $G_S^*(t)$ from the publicly observable dividend stream $D(t)$. To streamline notation for the information content of dividends, define $dI_0(t) := \big(\alpha_D\, D(t)\, dt + dD(t) - G_L(t)dt\big)/\sigma_D$ with $dB_0 := dB_D$ and

$$\tau_0 := \Omega\, \sigma_S^2/\sigma_D^2. \tag{6}$$

Then, the stochastic process

$$dI_0(t) := \tau_0^{1/2}\, \frac{G_S^*(t)}{\sigma_S\, \Omega^{1/2}}\, dt + dB_0(t) \tag{7}$$

is informationally equivalent to the dividend process $D(t)$ in equation (1). Assume it is common knowledge that the Brownian motions dB_0, dB_L, dB_S, dB_1,..., dB_N are independently distributed.

Let $\mathrm{E}_k^n\{\ldots\}$ denote the expectation of trader n calculated with respect to his beliefs about parameter values using information at time $t = kh$. This information consists of the history of public and private signals $dI_0(j)$ and $dI_n(j)$, $j \in [-\infty, t]$ and prices P_j, $j \leq k$, as discussed below. Let $G_{nS}(t) := \mathrm{E}_t^n\{G_S^*(t)\}$ and $G_{nS,k} := \mathrm{E}_k^n\{G_S^*(kh)\}$ denote trader n's estimate of the short-term growth rate at time t and time $t = kh$.

In equations (5) and (7), the parameter $\sigma_S\, \Omega^{1/2}$ is a scaling coefficient. Let Ω denote the steady state error variance of the estimate of $G_S^*(t)$, scaled in units of the standard deviation of its innovation σ_S:

$$\Omega := \mathrm{Var}\left\{\frac{G_S^*(t) - G_{nS}(t)}{\sigma_S}\right\}. \tag{8}$$

For example, if time is measured in years, then $\Omega = 4$ has the interpretation that the estimate of $G_S^*(t)$ is "behind" the actual value of $G_S^*(t)$ by an amount equivalent to four years of volatility unfolding at rate σ_S per year. Since Ω is constant in a steady state, it has no time argument.

The parameter τ_n is scaled in equations (5) and (7) so that $\tau_n dt$ is the R^2 of the predictive regression of the error $G_S^*(t) - G_{nS}(t)$ on $dI_n(t)$. It does not depend on the levels of the error variance Ω. The precision $\tau_n dt$ also measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio, describing how fast the information flow generates a signal of a given level of statistical significance.

Traders agree on the precision τ_0 of public information in equation (7). Since each trader believes his own signal has high precision τ_H and others' signals have low precision τ_L, symmetry implies that traders agree on the total precision

$$\tau := \tau_0 + \tau_H + (N-1)\,\tau_L. \tag{9}$$

Each trader chooses an optimal consumption path $c_n(t)$ and optimal portfolio holdings at time $t = kh$, denoted $S_{n,k}$, to maximize an additively separable exponential utility function with risk aversion A and time preference ρ:

$$\mathrm{E}_k^n \left\{ \int_{t=kh}^{\infty} \mathrm{e}^{-\rho(t-kh)}\; U(c_n(t))\, dt \right\}. \tag{10}$$

Both $c_n(t)$ and $S_{n,k}$ are calculated using information available at $t = kh$. The optimization problem is complicated by the fact that consumption is chosen continuously while portfolio holdings change only at trading period $t = kh$. For analytical tractability, we simplify the problem slightly by assuming that when trading occurs, traders choose both portfolio holdings and a consumption budget which does not change between rounds of trading. This makes the assumption that traders do not use new public information and their own new private information unfolding between trading rounds to adjust consumption between trading rounds. They cannot use other traders' private information between trading rounds because there are no updated prices from which to infer the average of other traders' private signals. As we shall see, traders have full information when they make decisions on quantities to trade.

The symmetric model structure is described by the following parameters: h, ρ, A, r, N, α_D, σ_D, α_L, σ_L, α_S, σ_S, τ_H, and τ_L. The model structure is common knowledge. Traders agree about all parameter values, except for symmetric disagreement about the precisions τ_H and τ_L of their own and other traders' signals. Symmetry implies that all traders agree about the variance Ω, precision of public information τ_0, and total precision τ.

Trade is generated by agreement to disagree about signal precision. Traders believe that they can make profits at the expense of others, even though it is common knowledge that aggregate profits are equal to zero.

Define $\breve{D}_{k+1}$ as the future value of dividends between rounds of trading:

$$\breve{D}_{k+1} = \mathrm{e}^{rh} \int_{kh}^{(k+1)h} \mathrm{e}^{-r(t-kh)}\, D(t)dt. \tag{11}$$

The optimization problem nests in a simple way and becomes the discrete-time problem

$$\max_{\substack{\{c_{n,j}\},j=k,k+1,\ldots,\infty \\ \{S_{n,j}\},j=k,k+1,\ldots,\infty}} \quad \mathrm{E}_k^n \sum_{j=k}^{\infty} \mathrm{e}^{-\rho(j-k)h}\, U_{n,j}(c_{n,j}), \tag{12}$$

subject to the budget constraint

$$W_{n,j+1} = \mathrm{e}^{rh} \left(W_{n,j} - hc_{n,j} - S_{n,j}P_j\right) + S_{n,j}\breve{D}_{j+1} + S_{n,j}P_{j+1}, \tag{13}$$

where $U_{n,j}(c_{nj})$ solves the continuous-time nested consumption subproblem

$$U_{n,j}(c_{n,j}) := \max_{c_n(jh+u),u\in[0,h]} \mathrm{E}_j^n \left\{ -\int_0^h \mathrm{e}^{-\rho u}\ \mathrm{e}^{-Ac_n(jh+u)}\ du \right\}, \tag{14}$$

subject to

$$hc_{n,j} = \int_0^h \mathrm{e}^{-ru}\, c_n(jh+u)du. \tag{15}$$

This optimization problem (12)–(15) is solved in Appendix A.2.

1.2. Model Solution

Stratonovich-Kalman-Bucy filtering implies that trader n's estimate $G_{nS,k}$ of the short-term growth rate at period k can be conveniently written as the weighted sum of three sufficient statistics or signals $H_{0,k}$, $H_{n,k}$, and $H_{-n,k}$, which summarize the information content of dividends, the trader's private information, and other traders' private information, respectively. Define

$$H_{n,k} := \int_{t=-\infty}^{kh} \mathrm{e}^{-(\alpha_S+\tau)\,(kh-t)}\ dI_n(t), \quad n = 0, 1, \ldots, N, \tag{16}$$

and

$$H_{-n,k} := \frac{1}{N-1} \sum_{\substack{m=1,..,N \\ m \neq n}} H_{m,k}. \tag{17}$$

These formulas have an intuitive interpretation. The signal $H_{n,k}$ is a sufficient statistic for trader n's own information. The average signal $H_{-n,k}$ is a sufficient statistic for other traders' information. The importance of each bit of information dI_n about the short-term growth rate decays exponentially at a rate $\alpha_S + \tau$, the sum of the natural decay rate α_S of the grow rate and the speed τ at which the others learn about it, defined in equation (9).

The filtering formulas imply that the steady-state error variance is given by

$$\Omega = \frac{1}{2\,\alpha_S + \tau}, \tag{18}$$

and trader n's expected growth rate at $t = kh$ is

$$G_{nS,k} := \sigma_S\, \Omega^{1/2} \left(\tau_0^{1/2}\, H_{0,k} + \tau_H^{1/2}\, H_{n,k} + (N-1)\, \tau_L^{1/2}\, H_{-n,k} \right). \tag{19}$$

When forming his estimate, each trader assigns a larger weight $\tau_H^{1/2}$ to his own signal $H_{n,k}$ and a smaller weight $\tau_L^{1/2}$ to each of the other traders' signals $H_{-n,k}$. Trade occurs as a result of the different weights used by traders in construction of their estimates.

Each trader calculates a target inventory proportional to his risk tolerance and the difference between his own valuation and the average valuation of other traders. The following theorem characterizes equilibrium for the continuous-time model with perfect competition.

THEOREM 1: *There exists a steady-state competitive equilibrium with symmetric linear strategies and with positive trading volume if and only if the three polynomial equations (A-45)–(A-47) have a solution, and traders' demand curves are downward sloping. Such an equilibrium has the following properties:*

1) *There is an endogenously determined constant $C_L > 0$, defined in equation (A-42), such that trader n's optimal inventories $S_{n,k}$ at period k are*

$$S_{n,k} = C_L\, (H_{n,k} - H_{-n,k}). \tag{20}$$

2) *There is an endogenously determined constant $C_G > 0$, defined in equation (A-40),*

such that the equilibrium price at period k is

$$P_k = \frac{D_k}{r+\alpha_D} + \frac{G_{L,k}}{(r+\alpha_D)(r+\alpha_L)} + C_G\,\frac{\bar{G}_{S,k}}{(r+\alpha_D)(r+\alpha_S)}, \tag{21}$$

where $G_{L,k}$ denotes the observable long-term growth rate and $\bar{G}_{S,k} := \frac{1}{N}\sum_{n=1}^{N} G_{nS,k}$ denotes the average of traders' expected short-term growth rates at time kh.

Theorem 1 implies that competitive traders immediately adjust inventories to levels equal to the target inventory $C_L\left(H_{n,k} - H_{-n,k}\right)$. The pricing formula (21) is similar to the average of traders' valuations (4) with one important exception. Averaging traders' expectations implies $C_G = 1$, but we will show below that $C_G < 1$ holds instead. The coefficient $C_G < 1$ makes the price different from the average valuations of all traders.

1.3. Price Dampening

In this section, we explain two mechanisms by which disagreement leads to price dampening: "static beliefs aggregation" and "dynamic beliefs aggregation."

Define the constant C_J as the ratio of the average of the square roots to the square root of the average of precisions:

$$C_J := \left(\tfrac{1}{N}\,\tau_H^{1/2} + \tfrac{N-1}{N}\,\tau_L^{1/2}\right)\,\left(\tfrac{1}{N}\,\tau_H + \tfrac{N-1}{N}\,\tau_L\right)^{-1/2}. \tag{22}$$

Now use equations (19) and (21) to write the price as

$$P_k = \frac{D_k}{r+\alpha_D} + \frac{G_{L,k}}{(r+\alpha_D)(r+\alpha_L)} + \frac{C_G\,\sigma_S\,\Omega^{1/2}}{(r+\alpha_D)(r+\alpha_S)}\left(\tau_0^{1/2}\,H_{0,k} + C_J\left(\tfrac{1}{N}\tau_H + \tfrac{N-1}{N}\tau_L\right)^{1/2}\sum_{n=1}^{N} H_{n,k}\right). \tag{23}$$

The constants C_G and C_J describe two related mechanisms for how different beliefs about the precisions of signals affect price.

In the benchmark case with $\tau_H = \tau_L$, equation (23) holds with $C_G = C_J = 1$, and the price describes a no-trade equilibrium in which all traders have the same beliefs and infer the same information from prices.

Thus, if all traders have average beliefs and assign the average precision $\frac{1}{N}\tau_H + \frac{N-1}{N}\tau_L$ symmetrically to all private signals, we obtain $C_G = C_J = 1$. When traders become relatively overconfident ($\tau_H > \tau_L$), holding the value of total precision $\tau_H + (N-1)\tau_L$

constant, then the value of Ω does not change, the equilibrium price is obtained from equation (23) with C_G and C_J having values less than one, as we discuss below.

The endogenous parameter C_J reflects *static* beliefs aggregation. It describes how traders form expectations about the short-term growth rate on average. Since the price is fully revealing, traders have the same information, but their expectations are different because they have different beliefs. When traders are relatively overconfident ($\tau_H > \tau_L$), Jensen's inequality implies $C_J < 1$ because the price reflects a weighted average of the square roots of precisions and the square root function is concave.[2] As disagreement increases, Jensen's inequality also implies that C_J decreases. Thus, holding total precision $\tau_H + (N-1)\tau_L$ constant, more disagreement about signal precision, measured by τ_H/τ_L, leads to more dampening of the price, as averaging traders' valuations dampens the weight the market assigns to the average signal about short-term growth rate relative to the benchmark case without disagreement and with the same total precision. This static dampening effect of $C_J < 1$ shows up even in an analogous one-period model, as we discuss in online Appendix B.1.

The endogenous parameter C_G reflects *dynamic* beliefs aggregation. It describes how the equilibrium price weights traders' valuations of fundamentals. If $C_G = 1$, the price is a weighted average of traders' expectations with weights summing to exactly one. If $C_G < 1$, then price is a dampened weighted average because the weights sum to less than one. With trading at discrete intervals, extensive numerical investigation shows that $C_G < 1$ always holds when $\tau_H > \tau_L$.

Intuitively, the dynamic dampening effect $C_G < 1$ is the result of how short-term speculative trading affects dynamic beliefs aggregation based on endogenous disagreement about the dynamics of the short-term growth rate. Each trader disagrees with others about how to interpret private information. He expects others to correct their erroneous valuations in the short run, yet ultimately converge towards his own valuation in the long run. Each trader attempts to profit by trading ahead of others' anticipated valuation revisions, even if this means trading against his own long-term valuation in the short run.

Consistent with the intuition that $C_G < 1$ is associated with speculative trading on short-term opportunities, we confirm that $C_G \to 1$ holds as trading opportunities occur at less frequent intervals. Figure 1 illustrates how C_G, C_J, and C_L (in equation (20)) depend on

[2]Jensen's inequality also implies that $C_J < 1$ as long as $\tau_H \neq \tau_L$. However, if $\tau_H < \tau_L$, then we obtain $C_L < 0$ and price impact (measured by the coefficient of inventories $S_n(t)$ in the price function (45) in Section 3.3) is negative. The demand curve slopes the wrong way. This case is thus less appealing.

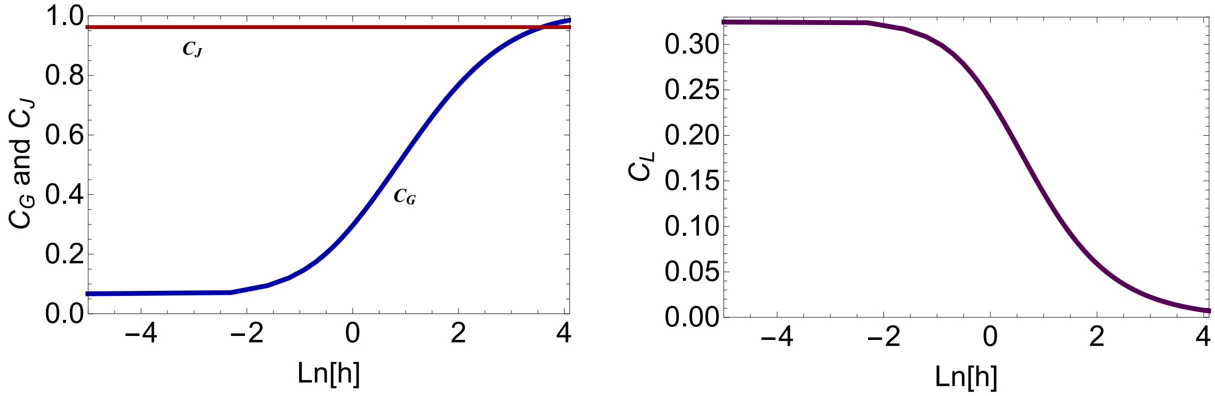


FIGURE 1. C_G, C_J, AND C_L AGAINST TIME INTERVAL $\ln(h)$.

the time interval between each trade h.[3] When the time interval h is large enough, C_L becomes small and $C_G \to 1$ holds; traders buy and hold small quantities. The dampening effect goes away because traders are not able to engage in short-term speculative trading based on disagreement about the short-term growth rate and thus $C_G \to 1$ holds. When the time interval h is close to zero, traders can trade aggressively against one another's perceived mistakes, the dampening effect converges to that of the continuous-time model. The constant C_J does not depend on the time interval between each trade h in both equation (22) and Figure 1. Figure 1 shows that both C_G and C_L become flat when $\ln(h) < -2$. This implies that the results of the discrete-time model converge to those of the continuous-time model approximately when $h < 0.135$ years, about seven weeks in this example.

Further intuition for the dynamic dampening effect is provided by Figure 2, which graphs buy-and-hold valuations $PV_n(0,t)$, $PV_{-n}(0,t)$, and $PV_p(0,t)$ with time t on the horizontal axis and the results of different present value calculations on the vertical axis (for $h = 0.1, 1, 10$, and 40).[4] Details of present-value calculations are given in equations (A-55), (A-57), and (A-59) in Appendix A.3. By assumption, these calculations are made using trader n's beliefs, but they are identical for all traders. For simplicity of exposition, we assume that the buy-and-hold valuations of all N traders coincide at time 0. Though the equilibrium price at time 0 is not equal to the average of these valuations due to the dynamic dampening effect of $C_G < 1$.

The horizontal solid line $PV_n(0,t)$ is based on the assumption that trader n liquidates the asset at date t at a valuation equal to his own estimate of its fundamental value.

[3]Parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_S = 0.02$, $\sigma_D = 0.5$, $\sigma_S = 0.1$, $\alpha_L = 0.02$, $\sigma_L = 0.1$, $\tau_0 = \Omega\sigma_S^2/\sigma_D^2 = 0.0054$, $\tau = 7.4$, $\tau_H = 1$, and $N = 100$.

[4]Parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_S = 0.02$, $\sigma_D = 0.5$, $\sigma_S = 0.1$, $\alpha_L = 0.02$, $\sigma_L = 0.1$, $\tau_0 = \Omega\sigma_S^2/\sigma_D^2 = 0.0054$, $\tau = 7.4$, $\tau_H = 1$, and $N = 100$, $G_{nS}(0) = G_{-nS}(0) = 0.08$, $G_L(0) = 0$, $D(0) = 0.7$.

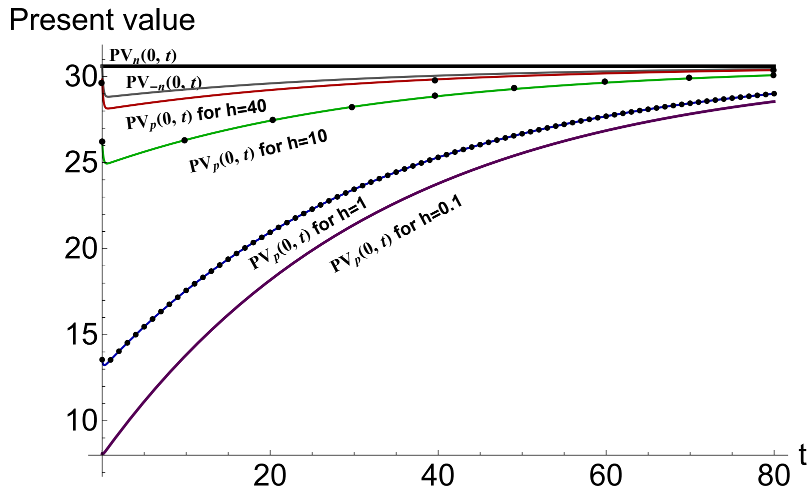


Figure 2. Present Value of Dividends and Liquidation Value from the Perspective of a Trader.

Since trader n applies Bayes law correctly given his beliefs, the martingale property of his valuation (law of iterated expectations) makes the present value $PV_n(0, t)$ a constant function for $t \geq 0$.

The curve $PV_{-n}(0, t)$ just below the line of $PV_n(0, t)$ depicts the present value of the asset based on the assumption that trader n liquidates the asset at a valuation equal to the average estimate of fundamental value of the other $N - 1$ traders. Due to disagreement about signal precision, trader n believes that the other $N - 1$ traders' estimates of the short-term growth rate $G_S^*(t)$ will mean revert to zero at rate $\alpha_S + \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2$, which is faster than the mean reversion rate α_S he assumes for his own forecast. Therefore, trader n believes that $PV_{-n}(0, t)$ will fall in the short run. Since he also believes that his own present value calculation is correct, he expects that $PV_{-n}(0, t)$ will rise back to his own estimate of the fundamental value in the long run, as illustrated in Figure 2.

The other four solid curves in Figure 2 are based on the assumption that trader n liquidates the asset at the equilibrium market price $P(t)$ for various time interval between trading $h = 0.1, 1, 10, 40$. Let $PV_p(0, t)$ label the graphs of these calculations in Figure 2. Consistent with the equilibrium result $0 < C_G < 1$, the initial price $P(0) := PV_p(0, 0)$ is lower than the consensus fundamental value, even if all traders by assumption agree about this current fundamental value. If prices were equal to the consensus fundamental valuation, all traders would want to hold short positions because all of them would expect prices to fall below fundamental value in the short run as the others learn about their mistakes and become temporarily bearish. As a result, the price $P(0)$ is dampened relative to the average fundamental valuation in the market. We refer to this mechanism as a Keynesian beauty contest since, in addition to disagreeing about the value of the asset at the present, traders agree to disagree about dynamics of their future valuations in the future and trade on this future disagreement at the present.

As we can see from Figure 2, the price is dampened more substantially if traders can trade more frequently (h is smaller). The dampening effect can push prices to much lower levels than trader's own valuation and other traders' valuations. If traders can only buy and hold ($h \to \infty$), then $C_G = 1$ holds and $PV_p(0,t)$ equals the weighted average valuation of $PV_n(0,t)$ and $PV_{-n}(0,t)$ with weights $1/N$ and $(N-1)/N$ respectively. If traders were not able to implement short-term strategies effectively due to long intervals between trading rounds, the profit opportunities could not be exploited and the dampening effect would go away. A formal analysis of expectations dynamics is provided in Appendix A.3.

The values of constants C_G and C_J also depend on the level of disagreement. Figure 3 illustrates that both C_J and C_G decrease when the degree of disagreement τ_H/τ_L increases while holding constant total precision ($h = 0.1, 1, 10, 40$).[5] Disagreement amplifies the dampening effect of $C_J < 1$ since it magnifies the effect of Jensen's inequality. Disagreement also leads to more pronounced price dampening ($C_G < 1$) due to the Keynesian beauty contest since traders have greater incentives to engage in short-term speculation.

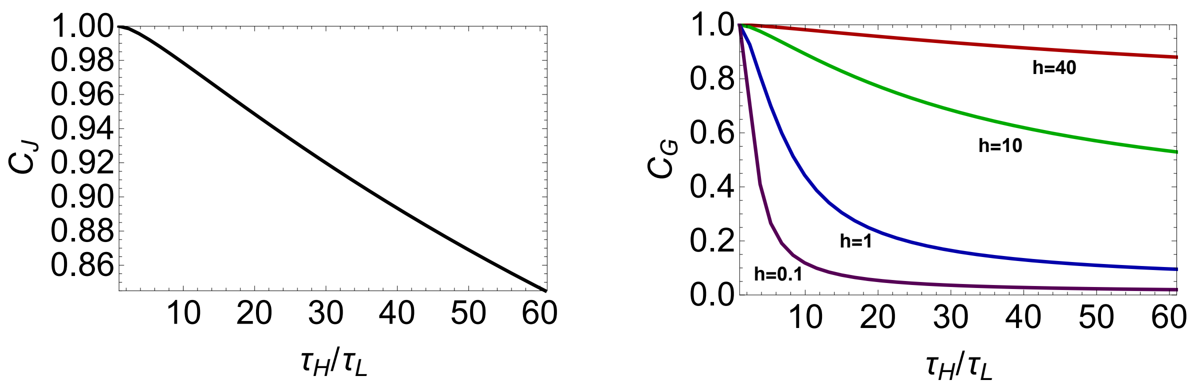


FIGURE 3. C_J AND C_G AGAINST τ_H/τ_L WHILE FIXING τ.

Since traders can trade frequently in active financial markets, we provide results for continuous trading ($h \to 0$). In this limiting case, we prove $C_G < 1$ analytically.

PROPOSITION 1 (Price Dampening with Continuous Trading): *Assume* $h \to 0$. *Rela-*

[5]Parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_S = 0.02$, $\sigma_D = 0.5$, $\sigma_S = 0.1$, $\alpha_L = 0.02$, $\sigma_L = 0.1$, $\tau_0 = \Omega\sigma_S^2/\sigma_D^2 = 0.0054$, $\tau = 7.4$, and $N = 100$.

tive overconfidence, $\tau_H > \tau_L$, implies price dampening:

$$(24) \qquad 0 < C_G \leq \left(1 + \frac{N-1}{N} \frac{\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2}{r + \alpha_S}\right)^{-1} < 1, \qquad \textit{and} \qquad 0 < C_J < 1.$$

A common prior, $\tau_H = \tau_L$, implies no price dampening: $C_G = 1$ and $C_J = 1$.

As disagreement τ_H/τ_L decreases, both constants converge to one and price dampening goes away. The proof is in Appendix A.4.

The following proposition describes a limiting case with a closed-form solution.

PROPOSITION 2 (Closed-Form Solution with $\tau_L = 0$): *Assume $h \to 0$, $\tau_L = 0$, $\tau_0 \to 0$, and $N \to \infty$. Then the three equations characterizing equilibrium, (A-45)–(A-47), have a closed-form solution presented in equations (A-66)–(A-68), implying $\lim_{N\to\infty} C_G = (r + \alpha_S)/(r + \alpha_S + \tau) < 1$, and $\lim_{N\to\infty} C_J = 0$.*

The proof is in Appendix A.5. In the limiting case with $N \to \infty$, we assume that $\tau_L = 0$ so that the total precision τ is fixed.[6] Proposition 2 implies that as the number of traders increases, C_J converges to zero and C_G converges to a constant limit which is less than one. Each trader believes that the other traders observe signals with no information and trade aggressively against one another's perceived mistakes. Price dampening is substantial.

Appendix A.6 proves that the risk tolerance parameter $1/A$ scales trading volume but has not effect on prices (including C_J and C_G).

2. Return Dynamics and Return Predictability

Next, we present the endogenously derived structural model for return dynamics and discuss its time-series properties in the context of our model.

It is difficult to design empirical tests if traders have heterogeneous beliefs because it is necessary to make a distinction between parameter values defined by traders' beliefs and parameter values that describe the empirically correct model. Since each trader believes his own private signal is more precise than other traders believe it to be, all traders' expectations cannot be correct simultaneously. Moreover, in a symmetric model with all

[6]If $\tau_L \neq 0$, then total precision $\tau \to \infty$ when $N \to \infty$. Both C_G and C_J would converge to one and there would be no dampening effect since each trader believes that the total precision of other $N-1$ private signal goes to infinity.

signals having the same empirically correct precision, none of the individual traders' beliefs could be correct.

For simplicity, we assume that a correct specification of the model has the same structure as traders believe but is described by different parameter values. We furthermore assume that the empirically correct precisions of all signals are the same.

Empirical model outcomes—such as conditional expected returns on the risky asset—depend on *both* the possibly incorrect parameters used by the traders *and* the empirically correct parameters. This makes expected returns complicated functions of the entire history of dividends and prices. Since the model converges to the continuous-time model as $h \to 0$, we describe our results for the continuous-time setting, which is more analytically tractable.

2.1. Inference under Empirically Correct Beliefs

We start by introducing empirically correct beliefs about model parameters. Let "hats" distinguish the empirically correct parameter values from the possibly incorrect beliefs of the traders.

Let precision $\hat{\tau}_0$ denote the empirically correct precision of public information. Let $\hat{\tau}_I$ denote the symmetric empirically correct precision of each private signal; for simplicity, all signals are assumed to have the same precision. As discussed below—using knowledge of both correct parameters and parameters used by traders—with continuous trading, the average signal across traders can be recovered from the histories of dividends and prices. The correct total precision is $\hat{\tau} = \hat{\tau}_0 + N\,\hat{\tau}_I$. From the perspective of each trader, the total precision is $\tau = \tau_0 + \tau_H + (N-1)\,\tau_L$. In general, these precisions are different ($\hat{\tau} \neq \tau$).

Except for beliefs about the parameters $\hat{\alpha}_L$, $\hat{\alpha}_S$, $\hat{\sigma}_S$, and $\hat{\tau}_I$, we assume that the empirically correct parameter values are the same as the parameter values used by traders. In particular, we assume that traders use correct parameters α_D, σ_D, and σ_L. Note that the value of σ_D can be estimated with perfect accuracy from observing quadratic variation in the dividend process $D(t)$ continuously, and the value of σ_L can be estimated with perfect accuracy from observing quadratic variation in $G_L(t)$ continuously. By placing "hats" over the variables, we obtain definitions of $\hat{\Omega}$, $\hat{\tau}_0$, $\hat{\tau}$, and $\hat{H}_n(t)$ for $n = 0, 1, \ldots, N$. In continuous time t, let $\hat{\mathrm{E}}_t\{\ldots\}$ denote the empirically correct expectation operator given all information at time t. The empirically correct unobserved short-term growth rate $G_S^*(t)$ follows the process

$$dG_S^*(t) := -\hat{\alpha}_S\, G_S^*(t)\, dt + \hat{\sigma}_S\, dB_S(t). \tag{25}$$

The market price aggregates the information content of the divided $D(t)$ and N signals $I_1(t), \ldots, I_N(t)$. The empirically correct long-term growth rate $G_L(t)$ follows the process

$$dG_L(t) := -\hat{\alpha}_L\, G_L(t)\, dt + \sigma_L\, dB_L(t). \tag{26}$$

Each signal $I_n(t)$ produces a continuous stream of information given by

$$dI_n(t) := \hat{\tau}_I^{1/2}\, \frac{G_S^*(t)}{\hat{\sigma}_S\, \hat{\Omega}^{1/2}}\, dt + d\hat{B}_n(t), \quad n = 1, \ldots, N, \tag{27}$$

where

$$d\hat{B}_n(t) = dB_n(t) + \left(\frac{\tau_n}{\sigma_S \Omega^{1/2}} - \frac{\hat{\tau}_n}{\hat{\sigma}_S \hat{\Omega}^{1/2}} \right) G_S^*(t) dt, \tag{28}$$

and dB_S, $d\hat{B}_1$, ...,$d\hat{B}_N$ are independent Brownian motions.

Define the dividend-information flow $dI_0(t)$ and its precision $\hat{\tau}_0$ as

$$dI_0(t) := \hat{\tau}_0^{1/2}\, \frac{G_S^*(t)}{\hat{\sigma}_S\, \hat{\Omega}^{1/2}}\, dt + dB_0(t), \qquad \text{with} \qquad \hat{\tau}_0 := \frac{\hat{\Omega}\, \hat{\sigma}_S^2}{\sigma_D^2}. \tag{29}$$

With a correct empirical specification, it is possible to solve a statistical inference problem similar to the one solved by traders and discussed in Section 1. The history of each information flow $I_n(t)$ can be summarized by a sufficient statistic $\hat{H}_n(t)$ defined as

$$\hat{H}_n(t) := \int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_S + \hat{\tau})\,(t-u)}\, dI_n(u), \quad n = 0, 1, \ldots, N. \tag{30}$$

Combining private signals and the public signal, define the aggregate sufficient statistic $\hat{H}(t)$ as the linear combination of $\hat{H}_0(t)$ and $\hat{H}_n(t)$, $n = 1, \ldots, N$, given by

$$\hat{H}(t) = \hat{\tau}_0^{1/2}\, \hat{H}_0(t) + \sum_{n=1}^{N} \hat{\tau}_I^{1/2}\, \hat{H}_n(t). \tag{31}$$

For comparison, we can define the continuous time statistics $H_n(t)$ and $H_{-n}(t)$ analogously to equations (16) and (17) and the market-implied aggregate sufficient statistic $H(t)$ using the market's implied precision weight:

$$(32) \qquad H(t) := \tau_0^{1/2}\, H_0(t) + \sum_{n=1}^{N} \tau_I^{1/2}\, H_n(t), \qquad \text{where} \qquad \tau_I^{1/2} := \frac{1}{N}\tau_H^{1/2} + \frac{N-1}{N}\tau_L^{1/2}.$$

Since the empirically correct model is symmetric, the statistic $\hat{H}(t)$ defined in (31) can be extracted from the history of public information (dividends, long-term growth rate, and market prices). Then the empirically correct estimate of the short-term growth rate $\hat{G}_S(t)$ can be written

$$(33) \qquad \hat{G}_S(t) := \hat{\mathrm{E}}\{G_S^*(t)\} = \hat{\sigma}_S\, \hat{\Omega}^{1/2}\, \hat{H}(t),$$

with steady-state error variance

$$(34) \qquad \hat{\Omega} := \hat{\mathrm{Var}}\left\{ \frac{G_S^*(t) - \hat{G}_S(t)}{\hat{\sigma}_S} \right\} = \frac{1}{2\,\hat{\alpha}_S + \hat{\tau}}.$$

As can be seen from equations (16) and (30), both sufficient statistics $\hat{H}_n(t)$ and $H_n(t)$ are linear combinations of increments in information flow, with weights decaying exponentially over time. The empirically correct decay rate may be different from the decay rate used by the traders. Therefore, in general we have

$$(35) \qquad \hat{\alpha}_S + \hat{\tau} \neq \alpha_S + \tau.$$

It can be shown that the sufficient statistics $\hat{H}_n(t)$ and $H_n(t)$, $n = 0, 1, \ldots, N$, relate to each other as follows,

$$(36) \qquad \hat{H}_n(t) = H_n(t) + (\alpha_S + \tau - \hat{\alpha}_S - \hat{\tau}) \int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_S + \hat{\tau})\,(t-u)}\; H_n(u)\, du.$$

If traders use the empirically correct mean-reversion rate ($\alpha_S = \hat{\alpha}_S$) and empirically correct total precision of the signals ($\tau = \hat{\tau}$), then we obtain $\hat{H}_n(t) = H_n(t)$. If traders have empirically incorrect beliefs about how quickly information decays, then the sufficient statistics $\hat{H}_n(t)$ and $H_n(t)$ are different, and the relationship between the two sufficient statistics depends on the entire history of information flow. For example, an empirically correct specification may assign higher weights to the information from the distant past if dividends are more persistent or signals are less precise than traders believe. In this case, we have $\alpha_S + \tau > \hat{\alpha}_S + \hat{\tau}$, and equation (36) shows how to obtain $\hat{H}_n(t)$ for trader n's signal as a function of the infinite history of a trader n's sufficient statistic $H_n(t)$.

We will be mostly interested in the aggregate statistic $\hat{H}(t)$ defined in (31). The histories of $H_0(t)$ and $H(t)$ can be recovered from the histories of dividends, prices, and the long-term growth rate. Equation (36) implies that $\hat{H}(t)$ can also be recovered from these histories. We will show next that the expected excess return has a specific closed form which depends on current and past prices and dividends as well as long-term growth rates.

2.2. Autocorrelation of the Holding-Period Excess Return

We describe return dynamics under empirically correct beliefs that the precision of the public signal is $\hat{\tau}_0$ and the precision of each private signal is $\hat{\tau}_I$. With continuous trading, equation (32) and the continuous version of equation (23) yield the continuous price $P(t)$ at time t:

$$P(t) = \frac{D(t)}{r + \alpha_D} + \frac{G_L(t)}{(r + \alpha_D)(r + \alpha_L)} + C_G \, \frac{\sigma_S \, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_S)} \, H(t). \tag{37}$$

The equilibrium return process has a linear structure[7]

$$dP(t) + D(t)\, dt - r\, P(t)\, dt = \big(b\, \hat{H}(t) - a\, H(t)\big)\, dt - \frac{\hat{\alpha}_L - \alpha_L}{(r + \alpha_D)(r + \alpha_L)} G_L(t) dt + d\hat{B}_r(t), \tag{38}$$

where a, b, and $d\hat{B}_r(t)$ are defined in equations (A-70), (A-71), and (A-72) in the Appendix. Equation (38) implies that the expected excess return is a linear combination of the average two dynamically changing statistics $H(t)$ and $\hat{H}(t)$ as well as the observable long-term growth rate $G_L(t)$.

We can also write the return dynamics (38) in a more intuitive and familiar form. Since the price $P(t)$ is a linear combination of $D(t)$, $G_L(t)$, and $H(t)$ from equation (37) and since

[7]Using equation (37), which expresses the market price $P(t)$ as a function of the dividend $D(t)$, the long-term growth rate $G_L(t)$, and the market's sufficient statistic $H(t)$, we can write an equation for $dP(t)$, plug in $dH_n(t)$ using equation (16), and plug in the correct empirical specification of the dynamics of $dI_n(t)$ from equation (27) and the correct estimate $\hat{G}_S(t)$ from equation (33).

$\hat{H}(t)$ can be recovered from the history of $H(t)$ using equation (36), both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$, dividends $D(t)$, and long-term dividend growth rate $G_L(t)$. This allows us to show that the return process, conditional on all public and private information, depends in a specific manner on the history of publicly observable dividends $D(t)$, prices $P(t)$, and $G_L(t)$.

THEOREM 2: *The equilibrium excess return dynamics can be expressed as a linear combination of past publicly observable dividends $D(t)$, prices $P(t)$, and long-term growth rates $G_L(t)$. More specifically, $dP(t) + D(t)\,dt - r\,P(t)\,dt$ can be written as*

$$
\begin{aligned}
&\alpha_1 \left(P(t) - \frac{D(t)}{r+\alpha_D} - \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)} \right) dt - \frac{\hat{\alpha}_L - \alpha_L}{(r+\alpha_D)(r+\alpha_L)} G_L(t) dt \\
&+ \alpha_2 \left(\int_{u=-\infty}^{t} \left(P(u) - \frac{D(u)}{r+\alpha_D} - \frac{G_L(u)}{(r+\alpha_D)(r+\alpha_L)} \right) \mathrm{e}^{-(\hat{\alpha}_S+\hat{\tau})(t-u)}\, du \right) dt \qquad (39) \\
&- \alpha_3 \left(\int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_S+\hat{\tau})\,(t-u)}\, dI_0(u) \right) dt + d\hat{B}_r(t),
\end{aligned}
$$

where the constants α_1, α_2, and α_3 are defined in equations (A-78), (A-79), and (A-80) in Appendix A.7 and $d\hat{B}_r(t)$, defined in equation (A-72), is a martingale increment with respect to information at time t.

The four terms in equation (39) capture different sources of return predictability. First, investors obtain a conditional excess return proportional to the deviation of the *current* price $P(t)$ from the unconditional valuation $D(t)/(r+\alpha_D) + G_L(t)/((r+\alpha_D)(r+\alpha_L))$. Second, a conditional excess return also relates to the long-term growth rate $G_L(t)$. It captures predictability related to traders using an incorrect mean reversion rate for long-term growth. Third, investors obtain a conditional excess return proportional to the *past* deviations of prices from the unconditional valuation and the *past* dividends surprises dI_0; the importance of each past component decays exponentially at rate $\hat{\alpha}_S + \hat{\tau}$.

The coefficients α_1, α_2, and α_3 are positive or negative depending on how the beliefs of traders about parameter values deviate from the empirically correct values of these parameters. When the information decay rate $\alpha_S+\tau$ used by traders is empirically correct, $\alpha_S + \tau = \hat{\alpha}_S + \hat{\tau}$, we obtain $\alpha_2 = 0$ but $\alpha_3 \neq 0$; except for dependence on past public information (likely empirically small), the expected return depends only on the current deviation of prices from unconditional values, not on past deviations.

Our model suggests that when $\alpha_S + \tau \neq \hat{\alpha}_S + \hat{\tau}$, it is the entire history of the dividend-to-price ratios and dividends—not only their current values—that should be included as explanatory variables in return-forecasting regressions in order to capture all information relevant for predicting the return. Thus, it may be warranted to consider more carefully VAR models with multiple lags, like Campbell and Shiller (1988), rather than a VAR model with one lag as is typical in more recent literature.

We next calculate the holding-period excess return. Let $R(t, t+T)$ denote the cumulative un-discounted holding-period mark-to-market cash flow per share on a fully levered investment in the risky asset from time t to time $t+T$:

$$R(t, t+T) = \int_{u=t}^{t+T} \Big(dP(u) + D(u)\, du - r\, P(u)\, du \Big). \tag{40}$$

From equation (38), we obtain

$$R(t, t+T) = \int_{u=t}^{t+T} \Big(b\, \hat{H}(u) - a\, H(u) \Big)\, du - \int_{u=t}^{t+T} \frac{\hat{\alpha}_L - \alpha_L}{(r+\alpha_D)(r+\alpha_L)} G_L(u) du + \int_{u=t}^{t+T} d\hat{B}_r(u). \tag{41}$$

Similar to equation (39), the holding-period excess return $R(t, t+T)$ can be expressed as a linear combination of past publicly observable dividends $D(t)$, prices $P(t)$, and long-term growth rate $G_L(t)$. The following theorem formally describes a structural model for holding-period excess returns.

THEOREM 3: *The holding-period excess return $R(t, t+T)$ can be represented as*

$$\begin{aligned} R(t, t+T) = {} & \beta_1(T) \left(P(t) - \frac{D(t)}{r+\alpha_D} - \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)} \right) - \frac{(\hat{\alpha}_L - \alpha_L)(1 - \mathrm{e}^{-\hat{\alpha}_L T})}{(r+\alpha_D)(r+\alpha_L)\hat{\alpha}_L} G_L(t) \\ & + \beta_2(T) \int_{u=-\infty}^{t} \left(P(u) - \frac{D(u)}{r+\alpha_D} - \frac{G_L(u)}{(r+\alpha_D)(r+\alpha_L)} \right) \mathrm{e}^{-(\hat{\alpha}_S + \hat{\tau})(t-u)}\, du \\ & - \beta_3(T) \int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_S + \hat{\tau})\,(t-u)}\, dI_0(u) + \bar{B}(t, t+T), \end{aligned} \tag{42}$$

where time-varying coefficients $\beta_1(T)$, $\beta_2(T)$, and $\beta_3(T)$ are defined in equations (A-89), (A-90), (A-91) in the Appendix and $\bar{B}(t, t+T)$ is a martingale increment defined in equation (A-86) in Appendix A.8.

Similar to Theorem 2, equation (42) implies that the holding-period excess returns depend on the deviation of the *current* price $P(t)$ from the unconditional valuation $D(t)/(r+\alpha_D)+$

$G_L(t)/((r+\alpha_D)(r+\alpha_L))$, the *past* deviations of prices from the unconditional valuation, the *past* dividends surprises dI_0, and a time-series value effect related to the long-term growth rate $G_L(t)$, as will be discussed below.

In the special case where traders use the empirically correct total precision of information flow and parameters describing the short-term growth rate ($\hat{\tau} = \tau$, $\hat{\alpha}_S = \alpha_S$, and $\hat{\sigma}_S = \sigma_S$), the correct precision of each signal is $\hat{\tau}_I = (\tau_H + (N-1)\tau_L)/N$, and we obtain the following proposition.

PROPOSITION 3: *If $\hat{\tau} = \tau$, $\hat{\alpha}_S = \alpha_S$, and $\hat{\sigma}_S = \sigma_S$, then $\beta_1(T) > 0, \beta_2(T) = 0$, $\beta_3(T) > 0$, and the expected holding-period excess return (42) can be written as*

$$\begin{aligned} \hat{\mathrm{E}}_t\{R(t,t+T)\} = {} & \beta_1(T)\Big(P(t) - \frac{D(t)}{r+\alpha_D} - \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)}\Big) \\ & - \frac{(\hat{\alpha}_L - \alpha_L)(1-\mathrm{e}^{-\hat{\alpha}_L T})}{(r+\alpha_D)(r+\alpha_L)\hat{\alpha}_L} G_L(t) - \beta_3(T) H_0(t). \end{aligned} \tag{43}$$

The coefficient $\beta_1(T)$ is positive and monotonically increasing in the horizon T.

The three terms on the right side of equation (43) are related to time-series momentum, time-series value, and overreaction to public information, respectively. Overall return dynamics is ultimately related to the relative magnitudes of these effects.

The positive coefficient $\beta_1(T)$ is related to *time-series momentum*. It is proportional to the difference between the current price $P(t)$ and the long-term unconditional valuation based on $D(t)$ and $G_L(t)$. This difference measures the dampened average of traders' expectations of the short-term growth rate $\bar{G}_S(t)$. Time series momentum is a consequence of underreaction of prices to $\bar{G}_S(t)$ resulting from both static and dynamic dampening effects, even when traders are correct on average. More specifically, in Appendix A.9, we show that the positive coefficient $\beta_1(T)$ can be decomposed into two terms, where the first term with $1 - C_G > 0$ results from the dynamic dampening effect of the Keynesian beauty contest and the second term with $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$ results from the static dampening effect of $C_J < 1$.[8]

The second term is related to a *time-series value* effect. It is proportional to the long-term growth rate $G_L(t)$. If traders think that the long-term growth rate mean-reverts at a slower rate than the empirically correct parameter ($\alpha_L < \hat{\alpha}_L$), then the proportionality

[8]Since traders are correct on average, we have $\hat{\tau}_I = \frac{1}{N}\tau_H + \frac{N-1}{N}\tau_L$. The definition of $\tau_I^{1/2}$ in equation (32) and $C_J < 1$ in equation (22) imply that $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$.

coefficient is negative. Stocks with high long-term growth rate $G_L(t)$ (growth stocks) tend to have lower returns than stocks with low long-term growth rates (value stocks). This time-series value effect implies mean-reversion in returns. The proportionality coefficient increases with time horizon, making this effect more pronounced in the long run. The third term is proportional to the public signal and the proportionality coefficient is negative. It also implies mean-reversion in returns because traders put too much weight on public signals relatively to the weight on private signals which is dampened due to the static dampening effect of $C_J < 1$. Prices tend to overreact to public information and this is corrected afterwards.

To formally examine return predictability, we derive the autocorrelations $\text{Corr}\{R(t - T_l), R(t + T_f)\}$ of the cumulative return for different lags T_l and leads T_f. The details are presented in equations (A-106), (A-113), and (A-114) in Appendix A.10. In this section for simplicity, we study correlations for equal leads and lags. We will use different lags and leads for the model calibration in Section 3.

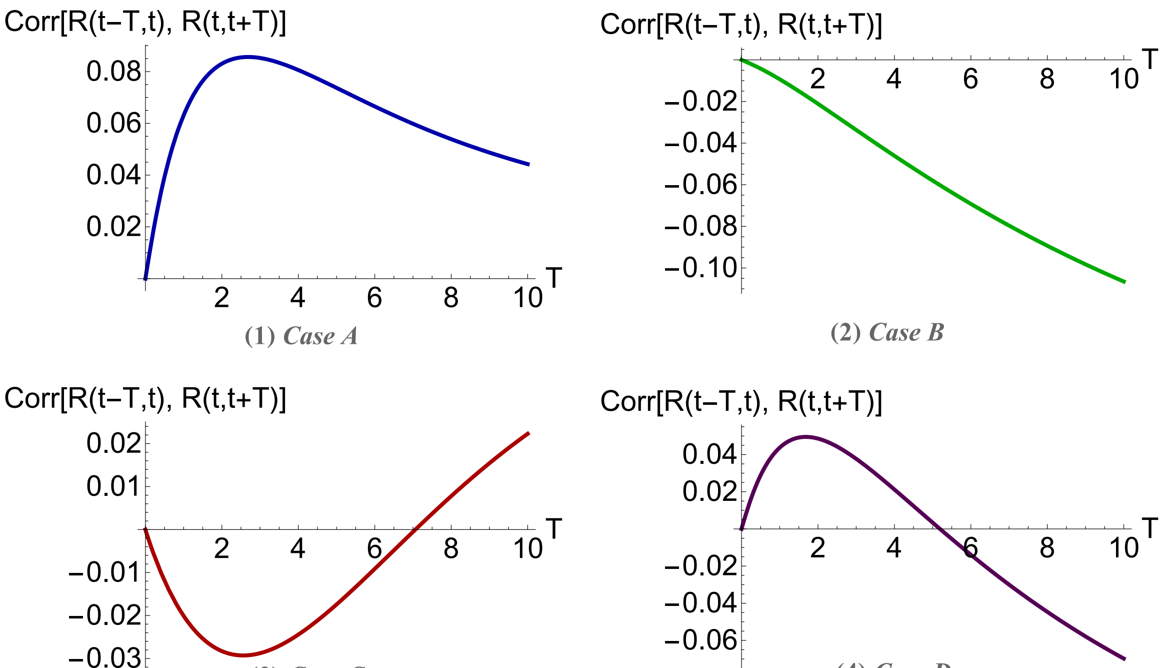
To develop the intuition that the holding-period return correlations depend on parameter values, we analyze and provide examples for four specific combinations of different parameter values (cases A, B, C, and D). In all four cases, we assume that traders' beliefs are "correct on average" (relatively overconfident but not absolutely overconfident). The traders also use correct parameters to describe the short-term growth rate. More specially, we focus on cases with $\hat{\tau} = \tau$, $\hat{\alpha}_S = \alpha_S$, and $\hat{\sigma}_S = \sigma_S$. For the more general cases in which traders use empirically incorrect parameter values such as $\hat{\alpha}_S$, $\hat{\sigma}_S$, and $\hat{\tau}$, a wide range of return patterns also arise in the equilibrium. We illustrate these patterns analytically and provide several numerical examples in online Appendix B.2.

Case A. Panel (1) of Figure 4 plots the autocorrelations $\text{Corr}\{R(t - T, t), R(t, t + T)\}$ of the cumulative excess returns for different horizons T when traders use the empirically correct mean-reversion rate of the long-term growth rate $\hat{\alpha}_L = \alpha_L$.[9] Proposition 3 implies a tendency for time-series momentum in returns since the coefficient $\beta_1(T)$ is positive and monotonically increases in horizon T.

Time-series momentum occurs due to price dampening. In Appendix A.10, we show analytically that the autocorrelation is positive for the limiting case studied in section 1.3 with $\tau_L = 0$, $\tau_0 \to 0$, and $N \to \infty$. Our extensive numerical analysis shows that the

[9] The parameters are $r = 0.01$, $N = 100$, $A = 1$, $\alpha_D = 0.1$, $\sigma_D = 0.5$, $\alpha_S = 0.4$, $\sigma_S = 1$, $\hat{\alpha}_L = \alpha_L = 0.008$, $\sigma_L = 0.06$, $\tau_H = 2$, $\tau_L = 0.02$.

autocorrelation is positive for a large range of parameter values for Case A. The positive autocorrelation implies that the conditional expected return $\hat{\mathrm{E}}\{R(t,t+T) \mid R(t-T,t)\}$ is increasing in $R(t-T,t)$, thus indicating time-series momentum in the sense that higher returns in the past tend to be followed by higher returns in the future.



(1) ***Case A*** **(2)** ***Case B*** **(3)** ***Case C*** **(4)** ***Case D***

Figure 4. The four panels illustrate different patterns of positive and negative correlation, $\mathrm{Corr}\{R(t-T,t), R(t,t+T)\}$, obtained over different holding periods T, when traders' beliefs are correct on average ($\tau = \hat{\tau}$, $\hat{\alpha}_S = \alpha_S$, $\hat{\sigma}_S = \sigma_S$).

Case B. Panel (2) of Figure 4 illustrates the correlation of the cumulative returns for the case when traders think that the long-term growth rate is more persistent than it actually is ($\alpha_L < \hat{\alpha}_L$).[10] Proposition 3 implies that if $\alpha_L < \hat{\alpha}_L$, then the second term of equation (43) leads to the time-series value effect and mean-reversion of returns. As illustrated in panel (2) of figure 4, if we set the disagreement level to be low, then the price dampening effect is small. Time-series value effect then dominates time-series momentum and makes autocorrelation in returns negative.

Case C. Panel (3) of Figure 4 illustrates the case when traders think that the long-term growth rate is more persistent than it actually is ($\alpha_L < \hat{\alpha}_L$), but the disagreement level is

[10]The parameters are the same as panel (1) in Figure 4, except $\tau_H = 0.1$ and $\hat{\alpha}_L = 0.05$.

higher and difference of $\hat{\alpha}_L - \alpha_L$ is larger than in Case B.[11] The expected holding period return exhibits short-run reversal followed by long-run momentum.

Case D. Panel (4) of Figure 4 illustrates a more realistic case when the expected holding-period excess returns first exhibit time-series momentum due to the price dampening effect and then mean reversion due to a time-series value effect, as explained after Proposition 3. The only difference from Case A is that we assume that $\hat{\alpha}_L > \alpha_L$.[12]

As our examples show, the expected returns exhibit different patterns depending on the parameter values. Figure 4 illustrates four possible patterns: (1) only momentum, (2) only mean-reversion, (3) first mean-reversion and then momentum, (4) first momentum and then mean-reversion.

In general, we find that the expected holding-period return patterns are highly sensitive to the parameter values. The last case is by-and-large consistent with empirical findings of short-run momentum and long-run mean-reversion. We will next calibrate our model parameters.

3. Model Calibration and Empirical Analysis

In this section, we first calibrate the parameter values of our structural model to generate quantitatively realistic patterns of returns correlation. We then test the new prediction of our model that time-series momentum tends to be stronger for stocks with greater trading volume.

3.1. Model Calibration

To calibrate the model parameters, we first conduct empirical analysis similar to Lee and Swaminathan (2000) (Column 1 of Table VIII). The sample consists of common stocks listed on the NYSE and AMEX during the period January 1965 through December 2006 with at least two years of data prior to the analysis.[13] We eliminate companies incorporated outside the United States, Americus Trust Components (Primes and Scores), closed-end funds, and real estate investment trusts.

Column (A) of Table 1 reports the time-series average of slope coefficients estimated from

[11]The parameters are the same as panel (1) in Figure 4, except $\alpha_L = 0.05$, $\alpha_S = 0.12$, $\sigma_S = 0.12$, and $\hat{\alpha}_L = 0.2$.

[12]The parameters are the same as panel (1) in Figure 4, except $\hat{\alpha}_L = 0.05$.

[13]We conduct our analysis using the sample period of 1965–2006 to exclude the financial crisis period.

monthly Fama–MacBeth cross-sectional regressions of the following model:

$$\text{Model } (A): \quad Return_{t+T_f-1,t+T_f,i} = a_{T_f,1} + b_{T_f,1} Return_{t-1,t,i} + e_{t+T_f-1,t+T_f,i},$$

where subscript i refers to stock i, $Return_{t+T_f-1,t+T_f,i}$ is the annual return T_f years ahead, and $Return_{t-1,t,i}$ is the annual return of the previous year, where a lead $T_f = 1, 2, 3, 4,$ or 5 years. The cross-sectional regression is run monthly using all stocks available. The standard errors of the time-series means are computed using the Hansen and Hodrick (1980) correction.

Column (A) of Table 1 reports that the slope coefficient is positive and significant for $T_f = 1$, negative and insignificant for $T_f = 2, 3, 4$, and negative and significant for $T_f = 5$. This confirms the presence of time-series momentum in Year 1 and significant reversal in Year 5. Our results are similar to the results in Lee and Swaminathan (2000).

	Time-Series Average Slope Coefficients, $b_{T_f,1}$	
T_f	Model (A)	Model (B)
1	0.0463 (2.9338)	0.0463 (2.9338)
2	-0.0302 (-1.6528)	0.0109 (0.3326)
3	-0.0190 (-1.0088)	-0.0110 (-0.2509)
4	-0.0265 (-1.5759)	-0.0549 (-1.1776)
5	-0.0366 (-2.1726)	-0.1076 (-1.5957)

Table 1—Regression Tests of Return Momentum and Reversal.

This table reports time-series average of slope coefficients estimated from monthly Fama–MacBeth cross-sectional regressions of models (A) and (B) run from January 1965 to December 2001. The coefficients are the time-series (444 months) means with t-statistics in parentheses.

Column (B) of Table 1 reports the time-series averages of slope coefficients from monthly

Fama–MacBeth cross-sectional regressions:

$$\text{Model } (B): \quad Return_{t,t+T_f,i} = a_{T_f,1} + b_{T_f,1} Return_{t-1,t,i} + e_{t,t+T_f,i},$$

where $Return_{t,t+T_f,i}$ is the cumulative return from time t to time $t+T_f$, and $Return_{t-1,t,i}$ is the annual return of the previous year, where $T_f = 1, 2, 3, 4, 5$ years. The patterns suggest similar returns dynamics.

To calibrate model parameters, we match theoretical regression coefficients of regressing the cumulative return $R(t, t+T_f)$ on $R(t-T_l, t)$, as in equation (A-115) in Appendix A.10, to the corresponding empirical values in column (B) of Table 1 for different T_f and $T_l = 1$.

We assume that traders are correct on average; they agree about the total precision of the short-term growth rate as well as the mean-reversion rate and instantaneous volatility of the short-term growth rate ($\hat{\tau} = \tau$, $\alpha_S = \hat{\alpha}_S$, $\sigma_S = \hat{\sigma}_S$). There remain twelve parameters: r, A, N, α_D, σ_D, α_S, σ_S, α_L, σ_L, $\hat{\alpha}_L$, τ_H, and τ_L. The magnitude and horizon of return momentum are determined by the level of disagreement τ_H/τ_L, the decay rate of signals about the short-term growth rate $\alpha_S + \tau$, and the size of σ_S. The magnitude and horizon of long run return reversal are determined by the difference between traders' beliefs and empirically correct beliefs about the mean-reversion rate of the long-term growth rate $\alpha_L - \hat{\alpha}_L$ and the size of σ_L.

To reduce the number of parameters to be estimated, we assume $r = 0.01$, $A = 1$, $N = 100$, $\alpha_D = 0.005$, $\sigma_D = 0.5$, $\tau_H = 10$, and $\tau_L = 0.1$. We then estimate five parameters α_S, σ_S, α_L, σ_L, and $\hat{\alpha}_L$ to match the regression coefficients to their empirical estimates.[14] To examine whether overconfidence can generate time-series momentum similar to what is observed empirically, the assumptions $N = 100$, $\tau_H = 10$, and $\tau_L = 0.1$ imply that each trader is extremely overconfident. The combined precision of 19.9 means that traders' inventories have a half-life of one or two weeks.[15] This implies that traders in the model engage in very short-term trading using their own signals to trade against others' signals in a Keynesian beauty contest. The estimated parameter values are reported in the last five rows of Table 2.

Figure 5 shows that our calibrated model closely matches the empirical estimates. The solid curve plots the theoretical regression coefficients $b_{T_f,1}$ for the calibrated parameter

[14] We run regression model (B) for $T_f = 0.5$, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, and 5 to obtain more observations for calibrating the parameter values of the model.

[15] Equation (20) in Theorem 1, equations (A-5) and (A-6) imply that traders' optimal inventories mean-revert at rate of $\alpha_S + \tau$.

Parameter	Description	Value
r	Risk-free rate	0.01
A	Risk aversion	1.00
α_D	Mean-reversion rate of dividend	0.04
σ_D	Instantaneous volatility of dividend	0.50
N	Numbers of traders	100.00
τ_H	Precision of trader n's signal	10.00
τ_L	Precision of others' signal	0.10
α_S	Mean-reversion rate of G_S	1.3130
σ_S	Instantaneous volatility of G_S	10.9792
α_L	Traders' mean-reversion rate of G_L	0.0014
$\hat{\alpha}_L$	Empirically correct mean-reversion rate of G_L	0.2043
σ_L	Instantaneous volatility of G_L	0.1169

Table 2—Parameter Calibration.

The top half of the table reports the given parameter values for r , A , N, α_D, σ_D, τ_H, and τ_L. The bottom half reports the estimated parameters α_S, σ_S, α_L, σ_L, and $\hat{\alpha}_L$.

values from Table 2. The dots correspond to the empirical slope estimates from regression model (B) in Table 1. The figure shows that the theoretical solid curve closely tracks all estimated dots.

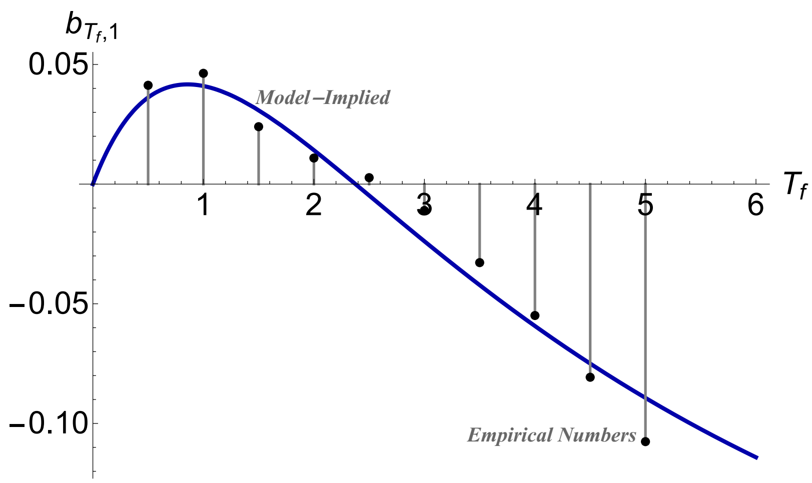


Figure 5. The figure shows theoretical regression coefficients (solid curve) and empirically estimated coefficients (dots) for different holding periods T_f.

The calibrated long-term mean reversion value $\alpha_L = 0.0014$ implies that traders believe that the long-term growth rate is very persistent. The calibrated correct long-term mean reversion rate $\hat{\alpha}_L = 0.2043$ implies that long-term growth $G_L(t)$ mean reverts after about $\ln(2)/0.2043 \approx 3.39$ years. This creates a time-series value anomaly matching the empirical finding that stock returns exhibit mean reversion over horizons of three to five years. The calibrated mean reversion rate for the short-term growth rate $\alpha_S = 1.3130$ suggests that $G_S(t)$ has a half-life of about $\ln(2)/1.3130 \approx 0.53$ years, again matching the empirical finding that momentum persists for only about six months or one year. The implied values of the two price dampening factors are $C_G = 0.1009$ and $C_J = 0.7727$. Return momentum arises from both pricing dampening effects.

3.2. Implications of Calibration for Trading Strategies

The calibration exercise provides practical insights into investment management. Using the calibrated parameter values in Table 2, we simulate 2000 independent histories of monthly excess return over 40 years based on (38). We then examine predictive regressions for one-month returns using several model specifications.[16]

When only the past six-month return $Return_{t-6m,t}$ is used to predict the next month's return, the coefficient is positive—indicating momentum—and the R-square is 0.0019. When a proxy for the value, $Value_t := P(t) - D(t)/(r + \alpha_D)$—measuring the arithmetic difference between market value $P(t)$ and "book value" $D(t)/(r + \alpha_D)$—is added, the R-square increases to 0.0109. As expected, the estimated coefficient on the value proxy is negative, indicating that value stocks tend to earn higher returns than growth stocks in our simulated data sample. The increase in R^2, from 0.0019 to 0.0109, indicates that a combination of value and momentum yields a much higher Sharpe ratio than momentum alone. When returns over the previous year, previous three years, and previous five years are added to the regression, the R^2 is 0.0120, only a modest increase from 0.0109.

The model assumes that traders in the market observe $G_L(t)$, which can be interpreted as a market consensus estimate of long-term growth. Given $G_L(t)$, the implied market estimate of short-term growth, $\bar{G}_S(t)$, can be inferred from prices. When the regression has only the two variables $G_L(t)$ and $\bar{G}_S(t)$, it is not surprising that the coefficient on $G_L(t)$ is negative and the coefficient on $\bar{G}_S(t)$ is positive. More surprisingly, the R^2 is 0.0334, much higher than 0.0109 or 0.0120. The calibrated model is telling us that much better investment performance can be obtained from a trading strategy which accurately

[16] We describe this procedure in more detail in online Appendix B.3. Table 1 in the Online Appendix reports the R-squares and estimates from these regressions.

distinguishes between long-term growth—to which the market overreacts—and short-term growth—to which the market underreacts—than using market-to-book ratios and past returns as proxies for value and momentum. Although the model assumes for simplicity that $G_L(t)$ and therefore $\bar{G}_S(t)$ are observable, the practical lesson might be that distinguishing between the two requires significant investment skill that is well-rewarded with superior performance.

Equipped with a calibrated structural model of returns dynamics, we next test some predictions of our model.

3.3. Model Predictions and Empirical Analysis

Time-series momentum exhibits cross-sectional patterns. Cremers and Pareek (2014) find momentum to be more substantial in stocks with more short-term trading. Moskowitz, Ooi and Pedersen (2012) find that more liquid contracts in equity-index, currency, commodity, and bond futures markets exhibit greater momentum. Zhang (2006) and Verardo (2009) find that momentum returns are larger for stocks with higher analysts' disagreement. Lee and Swaminathan (2000) note that momentum tends to be stronger for stocks with higher trading volume. The predictions of our model are consistent with these stylized facts.

To show this, we first construct several variables. Define trading volume as[17]

$$Volume := N\, \hat{\mathrm{E}} \left\{ \frac{|dS_n(t)|}{dt^{1/2}} \right\}. \tag{44}$$

Using equations (20) and (21), the equilibrium price can be written

$$P(t) = \frac{D(t)}{r + \alpha_D} + \lambda\, \frac{C_L}{\tau_I^{1/2}} \, \left(\tau_0^{1/2}\, H_0(t) + \tau_I^{1/2}\, N\, H_{-n}(t) \right) + \lambda\, S_n(t), \tag{45}$$

where λ is defined as

$$\lambda := \frac{C_G \sigma_S \Omega^{1/2} \tau_I^{1/2}}{(r + \alpha_D)(r + \alpha_S) C_L}. \tag{46}$$

The parameter λ can be interpreted as permanent price impact since it quantifies how accumulated inventories $S_n(t)$ affect the price. A smaller price impact parameter λ implies

[17] Although trading volume is theoretically infinite since inventories follow diffusions with continuous trading, scaling by $dt^{1/2}$ makes it finite and proportional to the standard deviations of changes in inventories over small time periods.

a deeper and more liquid market.

The variables $Volume$ and λ are both related to disagreement τ_H/τ_L. The market tends to be more liquid and trading volume tends to be higher when there is more disagreement. Using the calibrated parameter values of the model in Table 2 (varying τ_H and τ_L while holding total precision τ fixed), Figure 6 shows that market depth $1/\lambda$ increases in the degree of disagreement τ_H/τ_L since traders provide more liquidity to each other and hold larger positions. Trading volume tends to increase with disagreement as well.

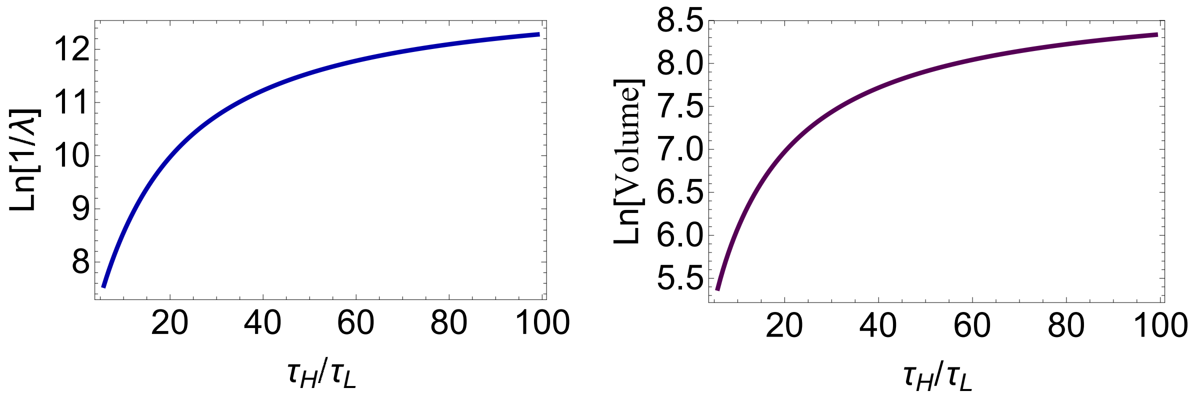


FIGURE 6. THE TWO PANELS PLOT $\ln(1/\lambda)$ AND $\ln(Volume)$ AGAINST τ_H/τ_L HOLDING τ FIXED.

Time-series momentum tends to be more pronounced when disagreement is larger. First, Figure 7 shows that the theoretical regression coefficient $b_{1/2,1/2}$ from regressing the cumulative return $R(t, t + 1/2)$ on the cumulative return $R(t - 1/2, t)$, as derived in equation (A-115) in Appendix A.10, increases in the degree of disagreement τ_H/τ_L.[18] Second, Figure 3 in Section 1.3 illustrates that both C_G and C_J decrease with disagreement, making price dampening more significant.

Combining the intuition from both figures, Figure 8 illustrates that time-series momentum tends to be stronger in the markets with more liquidity and trading volume.

We next examine empirically whether time-series momentum is more pronounced in stocks with higher trading volume. Following Campbell, Grossman and Wang (1993) and Lee and Swaminathan (2000), we use stock turnover as a proxy for trading volume.[19] For each month from January 1965 to December 2005, stocks are sorted based on monthly turnover averaged over the portfolio formation period, where monthly turnover is the ratio

[18]We set $T_l = T_f = 1/2$ (six months) in Figures 7 and 8. The regression coefficients b_{T_f,T_l} also increases in the degree of disagreement τ_H/τ_L for other parameter values of T_l and T_f.

[19]As noted in Lee and Swaminathan (2000), raw trading volume is unscaled and is likely to be highly correlated with firm size. In our model of one stock, the concept of trading volume corresponds to turnover.

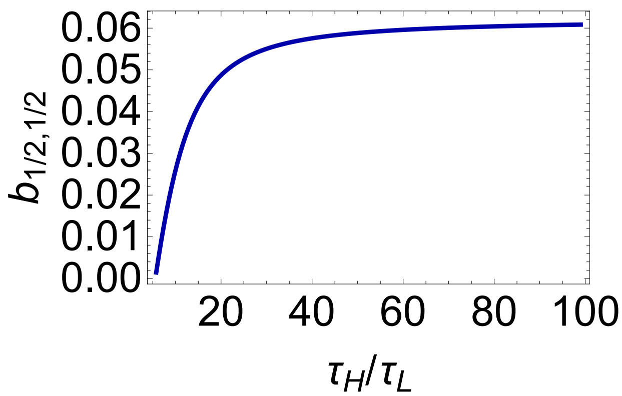


FIGURE 7. REGRESSION COEFFICIENT $b_{1/2,1/2}$ AGAINST τ_H/τ_L WHILE FIXING τ.

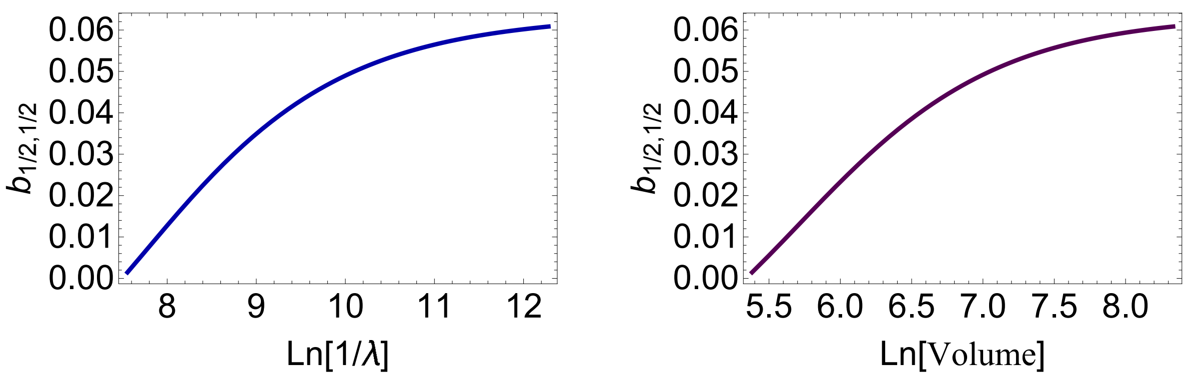


FIGURE 8. REGRESSION COEFFICIENT $b_{1/2,1/2}$ AGAINST $\ln(1/\lambda)$ AND $\ln(Volume)$ WHILE CHANGING τ_H/τ_L.

of the number of shares traded each month to the number of shares outstanding at the end of the month.

We focus on the stocks of the lowest quintile and highest quintile groups, based on turnover. Let subscript i refer to stock i. Define $Return_{t,t+1/2,i}$ as the six month cumulative return from time t to time $t+1/2$, define $Return_{t-1/2,t,i}$ as the six month cumulative return from time $t-1/2$ to time t, and let $Dummy_{t,i}$ denote a dummy variable, which is equal to one for stocks in the highest quintile and zero for stocks in the lowest quintile, based on turnover in the previous six months. We estimate the following two models using monthly Fama–MacBeth cross-sectional regressions:

$$\text{Model } (C): \quad Return_{t,t+1/2,i} = a_{1/2,1/2} + b_{1/2,1/2} Return_{t-1/2,t,i} + e_{t,t+1/2,i},$$

$$\text{Model } (D): \quad Return_{t,t+1/2,i} = a_{1/2,1/2} + b_{1/2,1/2} Return_{t-1/2,t,i}$$

$$+c_{1/2,1/2}Dummy_{t,i} + d_{1/2,1/2}Dummy_{t,i} \times Return_{t-1/2,t,i} + e_{t,t+1/2,i}.$$

	Time-Series Average Slope Coefficients	
Variable	Model (C)	Model (D)
$Return_{t-1/2,t,i}$	0.0426 (3.3724)	0.0127 (0.6454)
$Dummy_{t,i}$		-0.0297 (-2.5476)
$Return_{t-1/2,t,i} \times Dummy_{t,i}$		0.0450 (2.3581)

Table 3—Regression Tests of Return Momentum for the Stocks of the Lowest Quintile and Highest Quintile Based on Trading Volume (Turnover).

This table reports time-series average of slope coefficients estimated from monthly Fama–MacBeth cross-sectional regressions of models (C) and (D) run from January 1965 to December 2005 for the stocks of the lowest quintile and highest quintile based on turnover. The dummy variable ($Dummy_{t,i}$) is 1 for stocks in the highest turnover quintile and 0 for stocks in the lowest turnover quintile. The coefficients are the time-series (492 months) means with t-statistics in parentheses. The standard errors of the time-series means are computed using the Hansen and Hodrick (1980) correction.

Table 3 reports the slope estimates. Column (C) of Table 3 shows that the slope estimate is positive and significant, thus confirming the presence of short-run time-series momentum. Column (D) of Table 3 shows that momentum is more pronounced for stocks with substantial turnover. The coefficient on the interaction term $Return_{t-1/2,t,i} \times Dummy_{t,i}$ is positive and statistically significant at the 0.1% level, while the coefficient on $Return_{t-1/2,t,i}$ is no longer significant. The significantly negative coefficient of $Dummy_{t,i}$ is consistent with the empirical finding that lower volume (turnover) stocks generally have higher returns conditional on past returns (Lee and Swaminathan (2000)). The empirical results in Table 3 support the theoretical prediction that time-series momentum tends to be greater for stocks with higher trading volume (turnover).

4. Conclusion

We develop a dynamic model with discrete trading where traders with heterogeneous beliefs and private information solve complicated inference and optimization problems. Even though the prices fully reflect the average signal at each point in time and traders apply Bayes law consistently, traders regularly spot profit opportunities and think they can make money at the expense of others. The price is not equal to the average of traders' buy-and-hold valuations due to dampening from both static beliefs aggregation and dynamic beliefs aggregation. Returns are generally predictable. Different choices of model parameters generate time-series momentum and time-series mean reversion over different horizons.

Our structural model relates returns to the history of dividends, prices, and long-term growth rates. We calibrate the model parameter values and demonstrate that our model can generate quantitatively realistic return dynamics. Time-series momentum tends to be more pronounced when markets are more liquid and trading volume is more substantial.

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A. Proofs

A.1. The Dynamics of Key State Variables

In this subsection, we derive the conditional expectations of key state variables and the variance-covariance matrix at period k. Define the $N+1$ processes $dB_0^n(t)$, $dB_n^n(t)$, and $dB_m^n(t)$, $m = 1, \ldots, N$, $m \neq n$, by

$$dB_0^n(t) = \tau_0^{1/2} \, \frac{G_S^*(t) - G_{nS}(t)}{\sigma_S \, \Omega^{1/2}} \, dt + dB_D(t), \tag{A-1}$$

$$dB_n^n(t) = \tau_H^{1/2} \, \frac{G_S^*(t) - G_{nS}(t)}{\sigma_S \, \Omega^{1/2}} \, dt + dB_n(t), \tag{A-2}$$

(A-3)
$$dB^n_m(t) = \tau_L^{1/2}\,\frac{G^*_S(t) - G_{nS}(t)}{\sigma_S\,\Omega^{1/2}}\,dt + dB_m(t).$$

The superscript n indicates conditioning on beliefs of trader n. Since trader n's forecast of the error $G^*_S(t) - G_{nS}(t)$ is zero given his information set, these $N+1$ processes are independently distributed Brownian motions from the perspective of trader n. In terms of these Brownian motions, trader n believes that signals change as follows:

(A-4)
$$dH_0(t) = -(\alpha_S + \tau)\,H_0(t)\,dt + \tau_0^{1/2}\,\frac{G_{nS}(t)}{\sigma_S\,\Omega^{1/2}}\,dt + dB^n_0(t),$$

(A-5)
$$dH_n(t) = -(\alpha_S + \tau)\,H_n(t)\,dt + \tau_H^{1/2}\,\frac{G_{nS}(t)}{\sigma_S\,\Omega^{1/2}}\,dt + dB^n_n(t),$$

(A-6)
$$dH_{-n}(t) = -(\alpha_S + \tau)\,H_{-n}(t)\,dt + \tau_L^{1/2}\,\frac{G_{nS}(t)}{\sigma_S\,\Omega^{1/2}}\,dt + \frac{1}{N-1}\sum_{\substack{m=1\\ m\neq n}}^{N} dB^n_m(t).$$

Define

(A-7)
$$H^c_n(t) := H_n(t) + \hat{a}\,H_0(t), \quad H^c_{-n}(t) := H_{-n}(t) + \hat{a}\,H_0(t), \quad \hat{a} := \frac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}.$$

Using equations (A-4), (A-5), (A-6), and (A-7), write this as a continuous 4-vector stochastic process $y(t) = [D(t), G_L(t), H^c_n(t), H^c_{-n}(t)]'$ satisfying

(A-8)
$$dy(t) = K\,y(t)\,dt + C_z\,dZ(t),$$

where K is a 4×4 matrix and C_z is a 4×4 matrix given by

(A-9)
$$K = \begin{bmatrix} -\alpha_D & 1 & \sigma_S\Omega^{1/2}\tau_H^{1/2} & \sigma_S\Omega^{1/2}(N-1)\tau_L^{1/2} \\ 0 & -\alpha_L & 0 & 0 \\ 0 & 0 & -\alpha_S - \tau + \tau_H^{1/2}(\tau_H^{1/2} + \hat{a}\tau_0^{1/2}) & (N-1)\tau_L^{1/2}(\tau_H^{1/2} + \hat{a}\tau_0^{1/2}) \\ 0 & 0 & \tau_H^{1/2}(\tau_L^{1/2} + \hat{a}\tau_0^{1/2}) & -\alpha_S - \tau + (N-1)\tau_L^{1/2}(\tau_L^{1/2} + \hat{a}\tau_0^{1/2}) \end{bmatrix},$$

$$\text{(A-10)} \qquad C_z = \begin{bmatrix} \sigma_D & 0 & 0 & 0 \\ 0 & \sigma_L & 0 & 0 \\ \hat{a} & 0 & 1 & 0 \\ \hat{a} & 0 & 0 & \frac{1}{\sqrt{N-1}} \end{bmatrix},$$

and $dZ(t) = [dB_0^n(t), dB_L(t), dB_n^n(t), \frac{1}{\sqrt{N-1}} \sum_{\substack{m=1 \\ m\neq n}}^{N} dB_m^n(t)]'$ is a 4-dimensional Brownian motion.

Using results about linear continuous-time stochastic processes, we can represent the process $y_{k+1} = [D_{k+1}, G_{L,k+1}, H_{n,k+1}^c, H_{-n,k+1}^c]'$ in an integral form as

$$\text{(A-11)} \qquad y_{k+1} = \mathrm{e}^{Kh}\; y_k + \int_{kh}^{(k+1)h} \mathrm{e}^{K((k+1)h-t)}\; C_z\, dZ(t).$$

Equation (A-11) implies that

$$\text{(A-12)} \qquad \mathrm{E}_k^n\{y_{k+1}\} = \mathrm{e}^{Kh} \left[D_k, G_{L,k}, H_{n,k}^c, H_{-n,k}^c\right]',$$

$$\text{(A-13)} \qquad \mathrm{Var}_k\{y_{k+1}\} = \int_0^h \mathrm{e}^{K(h-t)}\, C_z C_z'\, \mathrm{e}^{K'(h-t)}\, dt.$$

We now derive $\mathrm{E}_k^n\{\breve{D}_{k+1}\}$, $\mathrm{Var}_k\{\breve{D}_{k+1}\}$, and $\mathrm{Cov}\{\breve{D}_{k+1}, y_{k+1}\}$. Define

$$\text{(A-14)} \qquad \breve{y}_{k+1} = \mathrm{e}^{rh} \int_{kh}^{(k+1)h} \mathrm{e}^{-r(t-kh)}\, y(t) dt.$$

It can be shown that

(A-15)

$$\breve{y}_{k+1} = (K-rI)^{-1}\Big(\left(\mathrm{e}^{(K-rI)h} - I\right) \mathrm{e}^{rh}\, y_k + \int_{kh}^{(k+1)h} \mathrm{e}^{K((k+1)h-t)}\, C_z dZ(t) - \int_{kh}^{(k+1)h} \mathrm{e}^{-r(t-(k+1)h)}\, C_z dZ(t)\Big).$$

Equation (A-15) implies that $\mathrm{E}_k^n\{\breve{D}_{k+1}\}$ is given by the first element in the 4×1 vector $\mathrm{e}^{rh}(K-rI)^{-1}\left(\mathrm{e}^{(K-rI)h} - I\right) y_k$. We can then derive

(A-16)

$$\mathrm{Cov}\{y_{k+1}, \breve{y}_{k+1}\} = \mathrm{Var}_k\{y_{k+1}\}(K' - rI)^{-1} - (K + rI)^{-1} \left(\mathrm{e}^{(K+rI)h} - I\right) C_z C_z' (K' - rI)^{-1},$$

(A-17)

$$\mathrm{Cov}\{\breve{y}_{k+1},\breve{y}_{k+1}\} = \Big((K-rI)^{-1}\,\mathrm{Var}_k\{y_{k+1}\} - (K-rI)^{-1}C_zC_z'(\mathrm{e}^{(K+rI)h}-I)'(K'+rI)^{-1}\Big)(K'-rI)^{-1} - \Big((K^2-r^2I)^{-1}(\mathrm{e}^{(K+rI)h}-I) + \frac{1-\mathrm{e}^{2rh}}{2r}(K-rI)^{-1}\Big)C_zC_z'(K'-rI)^{-1}.$$

Then $\mathrm{Var}_k\{\breve{D}_{k+1}\}$ is the (1,1) entry of the matrix $\mathrm{Cov}\{\breve{y}_{k+1},\breve{y}_{k+1}\}$ and $\mathrm{Cov}\{\breve{D}_{k+1},y_{k+1}\}$ is given by the first column of the matrix $\mathrm{Cov}\{y_{k+1},\breve{y}_{k+1}\}$.

A.2. Proof of Theorem 1

To solve the equilibrium, we conjecture that the price in period k is a linear function of D_k, $G_{L,k}$, and $\bar{G}_{S,k}$, of the form

(A-18)
$$P_k = \frac{D_k}{r+\alpha_D} + \frac{G_{L,k}}{(r+\alpha_D)(r+\alpha_L)} + C_G\,\frac{\bar{G}_{S,k}}{(r+\alpha_D)(r+\alpha_S)}.$$

Trader n's problem (10) can be rewritten in discrete-time form as (12) where $U_{n,j}$ is obtained by solving the maximization problem (14) subject to constraint (15). The first order condition yields

(A-19)
$$c(jh+t) = -\frac{1}{A}\left((\rho-r)t + \ln\frac{\lambda}{A}\right),$$

where λ is the Lagrange multiplier. By setting $t=0$ in equation (A-19), we get $c(jh) = -\frac{1}{A}\ln\frac{\lambda}{A}$. Therefore,

(A-20)
$$c(jh+t) = c(jh) + \frac{1}{A}(r-\rho)t.$$

Substituting (A-20) into constraint (15), we have

(A-21)
$$c(jh) = \frac{rh}{1-\mathrm{e}^{-rh}}c_{n,j} - \frac{r-\rho}{Ar(1-\mathrm{e}^{-rh})}\left(1-(rh+1)\,\mathrm{e}^{-rh}\right).$$

From equations (A-20), (A-21), and (14), we get

(A-22)
$$U_{n,j} = -h\exp\left(-A\frac{rh}{1-\mathrm{e}^{-rh}}c_{n,j}\right)\phi(r,\rho,h),$$

where $\phi(r,\rho,h)$ does not depend on $c_{n,j}$ and is defined as

(A-23)
$$\phi(r,\rho,h) = \frac{1-\mathrm{e}^{-rh}}{rh}\exp\left(\frac{r-\rho}{r(1-\mathrm{e}^{-rh})}\left(1-(1+rh)\,\mathrm{e}^{-rh}\right)\right) = 1-\frac{1}{2}\rho h + O(h^2).$$

Trader n's problem (12) is then equivalent to

$$\text{(A-24)}\qquad \max_{\substack{\{c_{n,j}\},j=k,k+1,\ldots,\infty\\ \{S_{n,j}\},j=k,k+1,\ldots,\infty}} \quad \mathrm{E}_k^n \sum_{j=k} -h\,\mathrm{e}^{-\rho(j-k)h} \exp\left(-A\frac{rh}{1-\mathrm{e}^{-rh}}c_{n,j}\right),$$

subject to the budget constraint (13). We conjecture and verify that the value function has the specific quadratic exponential form

(A-25)

$$V_k(W_{n,k}, H^c_{n,k}, H^c_{-n,k}) = -\exp\left(\psi_0 + \psi_W W_{n,k} + \tfrac{1}{2}\psi_{nn}(H^c_{n,k})^2 + \tfrac{1}{2}\psi_{xx}(H^c_{-n,k})^2 + \psi_{nx}H^c_{n,k}H^c_{-n,k}\right).$$

The five constants ψ_0, ψ_W, ψ_{nn}, ψ_{xx}, and ψ_{nx} have values consistent with a steady-state equilibrium. The terms ψ_{nn}, ψ_{xx}, and ψ_{nx} capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0.

Then the value function at period $k+1$ has the form

(A-26)

$$V_{k+1} = -\exp\left(\psi_0 + \psi_W W_{n,k+1} + \tfrac{1}{2}\psi_{nn}(H^c_{n,k+1})^2 + \tfrac{1}{2}\psi_{xx}(H^c_{-n,k+1})^2 + \psi_{nx}H^c_{n,k+1}H^c_{-n,k+1}\right).$$

The Hamilton-Jacobi-Bellman (HJB) equation for the discrete problem is

(A-27)

$$V_k(W_{n,k}, H^c_{n,k}, H^c_{-n,k}) = \max_{\substack{c_{n,k}\\ S_{n,k}}} \left\{-h\exp\left(-\frac{Arh}{1-\mathrm{e}^{-rh}}c_{n,k}\right) + \mathrm{e}^{-\rho h}\,\mathrm{E}_k^n\, V_{k+1}(W_{n,k+1}, H^c_{n,k+1}, H^c_{-n,k+1})\right\},$$

where the dynamics of wealth are given by (13) and the dynamics of $H^c_{n,k}$ and $H^c_{-n,k}$ can be obtained from equations (A-4), (A-5), (A-6), and (A-7). Define

$$\text{(A-28)}\qquad \begin{aligned} x_{k+1} :=& \left[D_{k+1}, \breve{D}_{k+1}, G_{L,k+1}, H^c_{n,k+1}, H^c_{-n,k+1}\right]' \\ &- \left[\mathrm{E}_k^n\, D_{k+1}, \mathrm{E}_k^n\, \breve{D}_{k+1}, \mathrm{E}_k^n\, G_{L,k+1}, \mathrm{E}_k^n\, H^c_{n,k+1}, \mathrm{E}_k^n\, H^c_{-n,k+1}\right]', \end{aligned}$$

where $\breve{D}_{k+1}$ is as defined in equation (11). Then $\mathrm{E}_k^n\, V_{k+1}$ can be obtained using

$$\text{(A-29)}\qquad \mathrm{E}_k^n\left\{\mathrm{e}^{-\alpha\left(\bar{A}+B'x+\frac{1}{2}x'Cx\right)}\right\} = \frac{1}{\sqrt{|I+\alpha C\Sigma|}}\,\mathrm{e}^{-\alpha\left(\bar{A}-\frac{1}{2}\alpha B'\Sigma(I+\alpha C\Sigma)^{-1}B\right)},$$

where x is an $n\times 1$ normal vector with mean zero and covariance matrix Σ, $\bar{A}$ is a scalar, B is an $n\times 1$ vector, C is an $n\times n$ symmetric matrix, I is the $n\times n$ identity matrix. The value of Σ can be obtained from Section A.1. More specifically, $\mathrm{Var}_k\{\breve{D}_{k+1}\}$ is the (1,1) entry of the matrix $\mathrm{Cov}\{\breve{y}_{k+1}, \breve{y}_{k+1}\}$ in equation (A-17), $\mathrm{Cov}\{\breve{D}_{k+1}, y_{k+1}\}$ is given by the first column of the matrix $\mathrm{Cov}\{y_{k+1}, \breve{y}_{k+1}\}$ in equation (A-16), and $\mathrm{Var}_k\{y_{k+1}\}$ is given in equation (A-13). Define α, $\bar{A}$, B, and C as

(A-30)

$$\begin{aligned}\bar{A} =& \psi_0 + \psi_W \,\mathrm{e}^{rh}\,(W_{n,k} - hc_{n,k} - S_{n,k}P_k) + \psi_W S_{n,k}\Bigg(\mathrm{E}_k^n\,\breve{D}_{k+1} + \frac{\mathrm{E}_k^n\,D_{k+1}}{r+\alpha_D} \\ &+ \frac{\mathrm{E}_k^n\,G_{L,k+1}}{(r+\alpha_D)(r+\alpha_L)} + \frac{C_G\sigma_S\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_S)}\left(\mathrm{E}_k^n\,H_{n,k+1}^c + (N-1)\,\mathrm{E}_k^n\,H_{-n,k+1}^c\right)\Bigg) \\ &+ \tfrac{1}{2}\psi_{nn}(\mathrm{E}_k^n\,H_{n,k+1}^c)^2 + \tfrac{1}{2}\psi_{xx}(\mathrm{E}_k^n\,H_{-n,k+1}^c)^2 + \psi_{nx}H_{n,k+1}^c H_{-n,k+1}^c,\end{aligned}$$

$$\text{(A-31)} \qquad \alpha = -1, \qquad B = \psi_W\psi_B S_{n,k} + \varphi_B, \qquad C = \begin{bmatrix} 0_{3\times3} & 0_{3\times2} \\ 0_{2\times3} & c_{2\times2} \end{bmatrix},$$

where

$$\text{(A-32)} \qquad c_{2\times2} = \begin{bmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{bmatrix}, \qquad \psi_B = \psi_{B1} + \frac{C_G\sigma_S\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_S)}\psi_{B2},$$

$$\text{(A-33)} \qquad \psi_{B1} = \left[\frac{1}{r+\alpha_D}, 1, \frac{1}{(r+\alpha_D)(r+\alpha_L)}, 0, 0\right]', \qquad \psi_{B2} = [0, 0, 0, 1, N-1]',$$

$$\text{(A-34)} \quad \varphi_B = \left[0, 0, 0,\ \psi_{nn}\,\mathrm{E}_k^n\,H_{n,k+1}^c + \psi_{nx}\,\mathrm{E}_k^n\,H_{-n,k+1}^c,\ \psi_{xx}\,\mathrm{E}_k^n\,H_{-n,k+1}^c + \psi_{nx}\,\mathrm{E}_k^n\,H_{n,k+1}^c\right].$$

Taking the first order condition with respect to $c_{n,k}$ in the HJB equation (A-27) yields

$$\text{(A-35)} \qquad c_{n,k}^* = -\frac{1-\mathrm{e}^{-rh}}{Arh}\left((r-\rho)h + \ln\left(\frac{\psi_W(1-\mathrm{e}^{-rh})}{Arh}\,\mathrm{E}_k^n\,V_{k+1}\right)\right).$$

Substituting (A-35) into the HJB equation (A-27), we obtain

$$\text{(A-36)} \qquad V_k = \mathrm{e}^{-\rho h}\left(1 - \frac{\psi_W(\mathrm{e}^{rh}-1)}{Ar}\right)\mathrm{E}_k^n\,V_{k+1}.$$

Taking the first order condition with respect to $S_{n,k}$ yields

$$\text{(A-37)} \qquad S_{n,k}^* = \frac{\mathrm{E}_k^n(P_{k+1} + \breve{D}_{k+1}) - \mathrm{e}^{rh}\,P_k + \varphi_B'\Sigma(I - C\Sigma)^{-1}\psi_B}{-\psi_W\psi_B'\Sigma(I - C\Sigma)^{-1}\psi_B}.$$

It can be shown that optimal trading strategy $S^*_{n,k}$ is a linear function of the state variables $H^c_{n,k}$ and $H^c_{-n,k}$.

Define constants c_{d1}, c_{d2}, c_{n1}, c_{x1}, c_{n2}, and c_{x2} by

$$\text{(A-38)}\quad \begin{aligned}
c_{d1} &:= \frac{\sigma_S \Omega^{1/2}\left(\mathrm{e}^{-\alpha_S h} - \mathrm{e}^{-\alpha_D h}\right)}{\alpha_D - \alpha_S}, \qquad c_{d2} := \frac{\sigma_S \Omega^{1/2}\left((r+\alpha_S)\,\mathrm{e}^{-(r+\alpha_D)h} - (r+\alpha_D)\,\mathrm{e}^{-(r+\alpha_S)h} + \alpha_D - \alpha_S\right)}{(\alpha_D-\alpha_S)(r+\alpha_D)(r+\alpha_S)},\\
c_{n1} &:= \frac{\mathrm{e}^{-(\alpha_S+\tau)h}\left(\mathrm{e}^{\tau h}\left(\tau_H^{1/2}\left(\tau_0+\tau_H\right) + (N-1)\tau_H\tau_L^{1/2}\right) + (N-1)\tau_L^{1/2}\left(\tau_0 + \tau_H^{1/2}\tau_L^{1/2} + (N-1)\tau_L\right)\right)}{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\tau},\\
c_{x1} &:= \frac{\mathrm{e}^{-(\alpha_S+\tau)h}\left(\mathrm{e}^{\tau h} - 1\right)(N-1)\tau_L^{1/2}\left(\tau_0 + (N-1)\tau_H^{1/2}\tau_L^{1/2} + \tau_H\right)}{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\tau},\\
c_{n2} &:= \frac{\mathrm{e}^{-(\alpha_S+\tau)h}\left(\mathrm{e}^{\tau h} - 1\right)\tau_H^{1/2}\left(\tau_0 + \tau_H^{1/2}\tau_L^{1/2} + (N-1)\tau_L\right)}{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\tau},\\
c_{x2} &:= \frac{\mathrm{e}^{-(\alpha_S+\tau)h}\left(\mathrm{e}^{\tau h}(N-1)\tau_L^{1/2}\left(\tau_0 + \tau_H^{1/2}\tau_L^{1/2} + (N-1)\tau_L\right) + \tau_H^{1/2}\left(\tau_0+\tau_H\right) + (N-1)\tau_H\tau_L^{1/2}\right)}{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\tau}.
\end{aligned}$$

Then φ_B, defined in equation (A-34), can be rewritten as

$$\text{(A-39)}\quad \begin{aligned}
\varphi_B &= \varphi_{B1} H^c_{n,k} + \varphi_{B2} H^c_{-n,k}, \ \text{where}\\
\varphi_{B1} &:= \left[0,0,0,\psi_{nn}c_{n1} + \psi_{nx}c_{n2}, \psi_{xx}c_{n2} + \psi_{nx}c_{n1}\right]',\\
\varphi_{B2} &:= \left[0,0,0,\psi_{nn}c_{x1} + \psi_{nx}c_{x2}, \psi_{xx}c_{x2} + \psi_{nx}c_{x1}\right]'.
\end{aligned}$$

The market clearing condition $\sum_{n=1}^N S^*_{n,k} = 0$ and equation (A-37) imply that

$$\text{(A-40)}\quad C_G = \frac{\mathrm{e}^{-\alpha_S h} - \mathrm{e}^{rh} - \frac{(r+\alpha_D)(r+\alpha_S)}{\sigma_S\Omega^{1/2}\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)}(\varphi'_{B1}+\varphi'_{B2})\Sigma(I-C\Sigma)^{-1}\psi_{B1}}{\frac{N\tau_0+(\tau_H^{1/2}+(N-1)\tau_L^{1/2})^2}{N\tau}\mathrm{e}^{-\alpha_S h} - \mathrm{e}^{rh} + \frac{(N-1)(\tau_H^{1/2}-\tau_L^{1/2})^2}{N\tau}\mathrm{e}^{-(\alpha_S+\tau)h} + \frac{1}{N}(\varphi'_{B1}+\varphi'_{B2})\Sigma(I-C\Sigma)^{-1}\psi_{B2}}.$$

Then, from equations (A-37) and (A-40), the optimal inventory for trader n is given by

$$\text{(A-41)}\quad S^*_{n,k} = C_L\left(H^c_{n,k} - H^c_{-n,k}\right),$$

where the constant C_L is defined as

$$\text{(A-42)}\quad \begin{aligned}
C_L = &\frac{1}{rAN\tau(r+\alpha_D)(r+\alpha_S)\psi'_B\Sigma(I-C\Sigma)^{-1}\psi_B}\Bigg(N\tau\left(\sigma_S\Omega^{1/2}\tau_H^{1/2}(\mathrm{e}^{rh} - \mathrm{e}^{-\alpha_S h}) + (r+\alpha_D)(r+\alpha_S)\varphi'_{B1}\Sigma(I-C\Sigma)^{-1}\psi_{B1}\right)\\
&+ C_G\sigma_S\Omega^{1/2}\Bigg(\tau\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\varphi'_{B1}\Sigma(I-C\Sigma)^{-1}\psi_{B2} + \mathrm{e}^{-\alpha_S h}\tau_H^{1/2}\left(N\tau_0 + \left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2\right)\\
&- \mathrm{e}^{rh}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\tau - \mathrm{e}^{-(\alpha_S+\tau)h}(N-1)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)\left(\tau_0 + \tau_H^{1/2}\tau_L^{1/2} + (N-1)\tau_L\right)\Bigg)\Bigg).
\end{aligned}$$

Equations (A-36) and (A-25) imply

$$\text{(A-43)}\quad \ln(-\mathrm{E}_k^n V_{k+1}) = \psi_0 + \psi_W W_{n,k} + \tfrac{1}{2}\psi_{nn}(H_{n,k}^c)^2 + \tfrac{1}{2}\psi_{xx}(H_{-n,k}^c)^2 + \psi_{nx} H_{n,k}^c H_{-n,k}^c + \rho h - \ln\left(1 - \frac{\psi_W(\mathrm{e}^{rh} - 1)}{Ar}\right).$$

Substituting (13), (A-12), (A-18), (A-26), (A-35), (A-41), and (A-43) into the HJB equation (A-27) and setting the constant term and the coefficients of $W_{n,k}$, $(H_{n,k}^c)^2$, $(H_{-n,k}^c)^2$, and $H_{n,k}^c\, H_{-n,k}^c$ to be zero, we obtain five equations, from which we can find five unknown parameters ψ_0, ψ_W, ψ_{nn}, ψ_{nx}, and ψ_{xx}.

By setting the constant term and coefficient of $W_{n,k}$ to be zero, we obtain

$$\text{(A-44)}\qquad \psi_W = -rA, \qquad \psi_0 = \frac{(r-\rho)h - (\mathrm{e}^{rh} - 1)\ln\frac{1-\mathrm{e}^{-rh}}{h} - \ln\sqrt{|I - C\Sigma|}}{\mathrm{e}^{rh} - 1}.$$

By setting the coefficients of $(H_{n,k}^c)^2$, $(H_{-n,k}^c)^2$ and $H_{n,k}^c\, H_{-n,k}^c$ to be zero, we obtain three polynomial equations in the three unknowns ψ_{nn}, ψ_{xx}, and ψ_{nx}. These three equations in three unknowns can be written as follows:

(A-45)

$$\begin{aligned}
0 = &- \tfrac{1}{2}\mathrm{e}^{rh}\,\psi_{nn} + rAC_LC_G\left(\mathrm{e}^{rh} - c_{n1} - (N-1)c_{n2}\right)\frac{\sigma_S\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_S)} \\
&- rAC_L\left(\mathrm{e}^{rh}\, c_{d2} + \frac{c_{d1}}{r+\alpha_D}\right)\tau_H^{1/2} + \tfrac{1}{2}\psi_{nn}c_{n1}^2 + \tfrac{1}{2}\psi_{xx}c_{n2}^2 + \psi_{nx}c_{n1}c_{n2} \\
&+ \tfrac{1}{2}r^2A^2C_L^2\psi_B'\Sigma(I - C\Sigma)^{-1}\psi_B - rAC_L\psi_B'\Sigma(I - C\Sigma)^{-1}\varphi_{B1} + \tfrac{1}{2}\varphi_{B1}'\Sigma(I - C\Sigma)^{-1}\varphi_{B1},
\end{aligned}$$

(A-46)

$$\begin{aligned}
0 = &- \tfrac{1}{2}\mathrm{e}^{rh}\,\psi_{xx} + rAC_LC_G\left(-\mathrm{e}^{rh}(N-1) + c_{x1} + (N-1)c_{x2}\right)\frac{\sigma_S\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_S)} \\
&+ rAC_L\left(\mathrm{e}^{rh}\, c_{d2} + \frac{c_{d1}}{r+\alpha_D}\right)(N-1)\tau_L^{1/2} + \tfrac{1}{2}\psi_{xx}c_{x2}^2 + \psi_{nx}c_{x1}c_{x2} \\
&+ \tfrac{1}{2}r^2A^2C_L^2\psi_B'\Sigma(I - C\Sigma)^{-1}\psi_B + rAC_L\psi_B'\Sigma(I - C\Sigma)^{-1}\varphi_{B2} + \tfrac{1}{2}\varphi_{B2}'\Sigma(I - C\Sigma)^{-1}\varphi_{B2},
\end{aligned}$$

(A-47)

$$\begin{aligned}
0 = &\,\mathrm{e}^{rh}\,\psi_{nx} + rAC_LC_G\left(\mathrm{e}^{rh}(N-2) - c_{x1} - (N-1)c_{x2} + c_{n1} + (N-1)c_{n2}\right)\frac{\sigma_S\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_S)} \\
&- rAC_L\left(\mathrm{e}^{rh}\, c_{d2} + \frac{c_{d1}}{r+\alpha_D}\right)\left((N-1)\tau_L^{1/2} - \tau_H^{1/2}\right) + \psi_{nn}c_{n1}c_{x1} + \psi_{xx}c_{n2}c_{x2} + \psi_{nx}(c_{n1}c_{x2} + c_{x1}c_{n2}) \\
&- r^2A^2C_L^2\psi_B'\Sigma(I - C\Sigma)^{-1}\psi_B - rAC_L\psi_B'\Sigma(I - C\Sigma)^{-1}(\varphi_{B2} - \varphi_{B1}) + \varphi_{B1}'\Sigma(I - C\Sigma)^{-1}\varphi_{B2}.
\end{aligned}$$

To summarize, optimal consumption is defined in (A-35), the optimal strategy is defined in (A-41), and the endogenous coefficient C_L is defined in (A-42). The equilibrium price is defined in (A-18), and the endogenous coefficient C_G is defined in (A-40). Parameters ψ_W and ψ_0 are presented in (A-44). Parameters ψ_{nn}, ψ_{nx}, ψ_{xx} are solved numerically from the system of the three equations (A-45)–(A-47). These results are presented in Theorem 1.

A.3. Dampening Effect: The Present Value of Expected Cumulative Dividends and Cash Flow

In this section, we discuss expectations of each trader about how his own valuation, the average valuation of other traders, and the market price evolve over time. The discussion here follows Kyle, Obizhaeva and Wang (2017), with some changes to accommodate differences between a setting with imperfect competition and the competitive setting of this paper. In line with Samuelson (1965), the trader's own valuation is a martingale with respect to the trader's own filtration. Each trader believes the average valuation of other traders and market prices follow a more complicated dynamics.

The coefficient $\tau_H^{1/2}$ in the second term on the right hand side of (A-5) is different from the coefficient $\tau_L^{1/2}$ in the second term on the right hand side of (A-6). This difference is the key driving force behind the price-dampening effect resulting from the Keynesian beauty contest. It captures the fact that—in addition to disagreeing about the value of the asset in the present—traders also disagree about the dynamics of their future valuations.

From equations (19), (A-4), (A-5), and (A-6), we can derive the stochastic process for $G_{nS}(t)$ and $G_{-nS}(t) := \frac{1}{N-1}\sum_{m=1,\ldots,N;\ m\neq n} G_{mS}(t)$ as follows:

$$\text{(A-48)}\quad dG_{nS}(t) = -\alpha_S\, G_{nS}(t)dt + \sigma_S\, \Omega^{1/2}\left(\tau_0^{1/2}dB_0^n(t) + \tau_H^{1/2}dB_n^n(t) + \tau_L^{1/2}\sum_{\substack{m=1\\ m\neq n}}^{N} dB_m^n(t)\right),$$

(A-49)

$$\begin{aligned} dG_{-nS}(t) = &-(\alpha_S + \tau)\, G_{-nS}(t)dt + \left(\tau_0 + \tau_L^{1/2}\left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2}\right)\right) G_{nS}(t)dt \\ &+ \sigma_S\, \Omega^{1/2}\left(\tau_0^{1/2}dB_0^n(t) + \tau_L^{1/2}dB_n^n(t) + \frac{\tau_H^{1/2} + (N-2)\tau_L^{1/2}}{N-1}\sum_{\substack{m=1\\ m\neq n}}^{N} dB_m^n(t)\right). \end{aligned}$$

From (A-49), when $G_{mS}(t) = G_{nS}(t)$, trader n believes that other traders' estimates of expected short-term growth rates $G_{mS}(t)$ will mean-revert to zero at a rate $\alpha_S + (\tau_H^{1/2} - \tau_L^{1/2})^2 > \alpha_S$. From (A-48), trader n believes that his own estimate of expected short-term growth rate $G_{nS}(t)$ will mean-revert to zero at a rate α_S.

From equations (1), (2), (3), (A-48), and (A-49), the expected dynamics of $G_{nS}(t)$, $G_{-nS}(t)$, $G_L(t)$, and $D(t)$ are given by

(A-50)
$$\mathrm{E}_0^n\{G_{nS}(t)\} = \mathrm{e}^{-\alpha_S t}\, G_{nS}(0),$$

(A-51)
$$\mathrm{E}_0^n\{G_{-nS}(t)\} = \frac{1}{\tau}\left(\tau_0 + \tau_L^{1/2}\left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2}\right)\right)\left(\mathrm{e}^{-\alpha_S t} - \mathrm{e}^{-(\alpha_S+\tau)t}\right) G_{nS}(0) + \mathrm{e}^{-(\alpha_S+\tau)t}\, G_{-nS}(0),$$

(A-52)
$$\mathrm{E}_0^n\{G_L(t)\} = \mathrm{e}^{-\alpha_L t}\, G_L(0),$$

(A-53)
$$\mathrm{E}_0^n\{D(t)\} = \frac{1}{\alpha_D - \alpha_S}\left(\mathrm{e}^{-\alpha_S t} - \mathrm{e}^{-\alpha_D t}\right) G_{nS}(0) + \frac{1}{\alpha_D - \alpha_L}\left(\mathrm{e}^{-\alpha_L t} - \mathrm{e}^{-\alpha_D t}\right) G_L(0) + \mathrm{e}^{-\alpha_D t}\, D(0).$$

The present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t, using trader n's estimate of fundamental value, is

(A-54)
$$PV_n(0,t) := \mathrm{E}_0^n\left\{\int_0^t \mathrm{e}^{-ru}\, D(u)du + \mathrm{e}^{-rt}\left(\frac{D(t)}{r+\alpha_D} + \frac{G_{nS}(t)}{(r+\alpha_D)(r+\alpha_S)} + \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)}\right)\right\}.$$

Substituting (A-50) and (A-53) into (A-54), it can be shown that (A-54) is equal to

(A-55)
$$PV_n(0,t) = F_n(0) = \frac{D(0)}{r+\alpha_D} + \frac{G_{nS}(0)}{(r+\alpha_D)(r+\alpha_S)} + \frac{G_L(0)}{(r+\alpha_D)(r+\alpha_L)}.$$

The present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t, using others' valuations, is

(A-56)
$$PV_{-n}(0,t) := \mathrm{E}_0^n\left\{\int_0^t \mathrm{e}^{-ru}\, D(u)du + \mathrm{e}^{-rt}\left(\frac{D(t)}{r+\alpha_D} + \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)} + \frac{G_{-nS}(t)}{(r+\alpha_D)(r+\alpha_S)}\right)\right\}.$$

Assuming $G_{mS}(0) = G_{nS}(0) = \bar{G}(0)$ and substituting (A-50)–(A-53) into (A-56), it can be shown that equation (A-56) is equal to

(A-57)
$$PV_{-n}(0,t) = F_n(0) + \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2}{\tau(r+\alpha_S)(r+\alpha_D)}\left(\mathrm{e}^{-(r+\alpha_S+\tau)t} - \mathrm{e}^{-(r+\alpha_S)t}\right) G_{nS}(0).$$

Similarly, the present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t at the equilibrium price $P(t)$ is

(A-58)

$$PV_p(0,t) := \mathrm{E}_0^n \left\{ \int_0^t \mathrm{e}^{-ru}\, D(u)du + \mathrm{e}^{-rt} \Big(\frac{D(t)}{r+\alpha_D} + \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)} + \frac{C_G\, \bar{G}_S(t)}{(r+\alpha_D)(r+\alpha_S)} \Big) \right\}.$$

Substituting (A-50)–(A-53) into (A-58), it can be shown that (A-58) is equivalent to

$$\begin{aligned} PV_p(0,t) =& F_n(0) + \frac{C_G \Big(N - (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} \, (N-1) \Big) - N}{N \, (r+\alpha_S) \, (r+\alpha_D)} \, \mathrm{e}^{-(r+\alpha_S)t} \; G_{nS}(0) \\ &+ \frac{C_G \, (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} \, (N-1)}{N \, (r+\alpha_S) \, (r+\alpha_D)} \, \mathrm{e}^{-(r+\alpha_S+\tau)t} \; G_{nS}(0). \end{aligned} \tag{A-59}$$

We next provide results which calculate the derivative of the present value of cash flows $PV_{-n}(0,t)$ with respect to time. From (A-56), it follows that

$$\frac{dPV_{-n}(0,t)}{dt} = \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2 \, G_{nS}(0) \; \mathrm{e}^{-(r+\alpha_S)t}}{\tau(r+\alpha_S)(r+\alpha_D)} \Big((r+\alpha_S) - (r+\alpha_S+\tau)\, \mathrm{e}^{-\tau t} \Big). \tag{A-60}$$

Equation (A-60) implies that $dPV_{-n}(0,t)/dt < 0$ iff $t < \frac{1}{\tau} \ln \Big(1 + \frac{\tau}{r+\alpha_S} \Big)$.

We now calculate the derivative of the present value of cash flows $PV_p(0,t)$ with respect to time. From (A-59), it follows that

(A-61)

$$\begin{aligned} \frac{dPV_p(0,t)}{dt} = \frac{G_{nS}(0)\, \mathrm{e}^{-(r+\alpha_S)t}}{N \, (r+\alpha_S) \, (r+\alpha_D)} \Bigg(\Big(N - C_G \Big(N - (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N-1) \Big) \Big) (r+\alpha_S) \\ C_G \, (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N-1)(r+\alpha_S+\tau)\, \mathrm{e}^{-\tau t} \Bigg). \end{aligned}$$

Clearly, (A-61) implies $dPV_p(0,t)/dt \to 0$ when $t \to \infty$. Define

$$\hat{t} := -\frac{1}{\tau} \ln \left(\left(1 + \frac{(1-C_G)N\tau}{C_G(\tau_H^{1/2} - \tau_L^{1/2})^2(N-1)} \right) \frac{r+\alpha_S}{r+\alpha_S+\tau} \right). \tag{A-62}$$

Equation (A-61) implies $dPV_p(0,t)/dt > 0$ if and only if $t > \hat{t}$. It can be shown that $\hat{t} > 0$ if and only if $C_G > \hat{C}_G := \Big(1 + (1 - 1/N)(\tau_H^{1/2} - \tau_L^{1/2})^2/(r+\alpha_S) \Big)^{-1}$. This further yields the following results:

- If $C_G \leq \hat{C}_G$, then $dPV_p(0,t)/dt > 0$ for all $t > 0$.

- If $C_G > \hat{C}_G$, then $dPV_p(0,t)/dt = 0$ for $t = \hat{t}$, $dPV_p(0,t)/dt > 0$ for $t > \hat{t}$, and $dPV_p(0,t)/dt < 0$ for $t < \hat{t}$.

From Proposition 1, $C_G \leq \hat{C}_G$ holds in the limiting case with $h = 0$; therefore, $PV_p(0,t)$ increases monotonically over time for $h = 0$.

A.4. Proof of Proposition 1

Assume $\tau_H > \tau_L$. Information cannot have negative value in the value function (A-25) since traders can ignore it. Therefore, the 2×2 matrix

$$\begin{bmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{bmatrix} \tag{A-63}$$

must be negative semi-definite. This implies $\psi_{nn} \leq 0$, $\psi_{xx} \leq 0$, and $\psi_{nx}^2 \leq \psi_{nn}\psi_{xx}$. It follows that $\psi_{nn} + \psi_{xx} + 2\psi_{nx} \leq 0$. In the continuous-time model ($h \to 0$), we can show (A-64)

$$C_G = \frac{N(r+\alpha_S)\left(\sigma_S \Omega^{1/2} + \sigma_D \hat{a}(\psi_{nn} + \psi_{xx} + 2\psi_{nx})/\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\right)}{\sigma_S \Omega^{1/2}\left(N(r+\alpha_S) + (N-1)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 - (1+N\hat{a}^2)(\psi_{nn} + \psi_{xx} + 2\psi_{nx})\right)}.$$

Then, from equation (A-64), we have

$$C_G \leq \left(1 + (1 - 1/N)(\tau_H^{1/2} - \tau_L^{1/2})^2/(r+\alpha_S)\right)^{-1} < 1. \tag{A-65}$$

Jensen's inequality implies $0 < C_J < 1$.

Assuming $\tau_H = \tau_L$, then clearly $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves the three equations (A-45)–(A-47), and additionally we get $C_G = 1$ and $C_L = 0$ from equations (A-40) and (A-42). There is no trading.

A.5. Proof of Proposition 2

We set $\tau_L = 0$ and $h \to 0$, and then evaluate the solution in the limit as $N \to \infty$ and $\hat{a} \to 0$. We conjecture and verify that $\psi_{nn} = \bar{\psi}_{nn}$, $\psi_{nx} = \bar{\psi}_{nx}$, and $\psi_{xx} = \bar{\psi}_{xx}$, where $\bar{\psi}_{nn}$, $\bar{\psi}_{nx}$, and $\bar{\psi}_{xx}$ are constants that do not depend on N.

Solving the system of equations (A-45)–(A-47) yields

$$\bar{\psi}_{nn} = \tfrac{1}{2}\left(r + 2(\alpha_S + \tau - \tau_H) - \left((r + 2(\alpha_S + \tau - \tau_H))^2 + \frac{4\Omega\sigma_S^2\tau_H}{\sigma_D^2}\right)^{1/2}\right), \tag{A-66}$$

(A-67)
$$\bar{\psi}_{nx} = \frac{\Omega \sigma_S^2 \tau_H / \sigma_D^2}{r + 2(\alpha_S + \tau) - \tau_H - \bar{\psi}_{nn}},$$

(A-68)
$$\bar{\psi}_{xx} = \frac{1}{r + 2\alpha_S + 2\tau} \left(\bar{\psi}_{nx}^2 - \frac{\Omega \sigma_S^2 \tau_H}{\sigma_D^2} \right).$$

Equations (A-64) and (A-42) imply

(A-69)
$$C_G \to \frac{r + \alpha_S}{r + \alpha_S + \tau} < 1, \qquad C_L = \frac{\Omega^{1/2} \sigma_S \tau_H^{1/2} (r + \alpha_D)}{A r \sigma_D^2}.$$

A.6. C_J, C_G, and Risk Aversion

The following proposition describes how C_J and C_G depend on risk aversion.

PROPOSITION 4: *The constants C_J and C_G do not depend on risk aversion A.*

It can be shown that parameters C_J and C_G remain the same when the risk aversion parameter A changes.

Let a vector $[\psi^*_{nn},\ \psi^*_{nx},\ \psi^*_{xx}]$ be a solution to the system (A-45)–(A-47) for exogenous parameters A, σ_D, σ_S, r, α_S, α_D, τ_0, τ_L, and τ_H. If risk aversion is rescaled by factor F from A to A/F and other exogenous parameters are kept unchanged, then it is straightforward to show that the vector $[\psi^*_{nn},\ \psi^*_{nx},\ \psi^*_{xx}]$ is still the solution to the system (A-45)–(A-47). From equations (46), (A-64), and (A-42), it then follows that C_L changes to $C_L F$, λ changes to λ/F, but C_G remains the same.

A.7. Proof of Theorem 2

Define a and b as

(A-70)
$$a := \frac{\sigma_S\, C_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_S)} \, (\alpha_S + r + \tau),$$

(A-71)
$$b := \frac{\hat{\sigma}_S\, \hat{\Omega}^{1/2}}{r + \alpha_D} + \frac{\sigma_S\, C_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_S)} \, \left(\tau_0^{1/2} \hat{\tau}_0^{1/2} + \tau_I^{1/2}\, N\, \hat{\tau}_I^{1/2} \right).$$

From direct calculation, the uncertainty term $d\hat{B}_r(t)$ in equation (38) is defined as

(A-72)
$$d\hat{B}_r(t) := \frac{\sigma_S\, C_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_S)} \left(\tau_0^{1/2}\, dB_0^*(t) + \tau_I^{1/2}\, N\, d\bar{B}^*(t) \right) + \frac{\sigma_D}{(r + \alpha_D)} dB_0^*(t) + \frac{\sigma_L dB_L(t)}{(r + \alpha_D)(r + \alpha_L)}.$$

The processes $d\bar{B}^*(t)$ and $dB_0^*(t)$, defined as

$$d\bar{B}^*(t) := \hat{\tau}_I^{1/2}\,(\hat{\sigma}_S\,\hat{\Omega}^{1/2})^{-1}\,(G_S^*(t) - \hat{G}_S(t))\,dt + \frac{1}{N}\sum_{n=1}^{N} d\hat{B}_n(t), \tag{A-73}$$

$$dB_0^*(t) := \hat{\tau}_0^{1/2}\,(\hat{\sigma}_S\,\hat{\Omega}^{1/2})^{-1}\,(G_S^*(t) - \hat{G}_S(t))\,dt + dB_0(t), \tag{A-74}$$

are Brownian motions under the empirically correct beliefs. Note that the variance of $dB_0^*(t)$ is equal to one, but the variance of $d\bar{B}^*(t)$ is equal to $1/N$ per unit of time. So the instantaneous variance of the excess return is given by

(A-75)

$$\hat{\mathrm{Var}}\left\{\frac{d\hat{B}_r(t)}{dt^{1/2}}\right\} = \Big(\frac{\sigma_D}{r+\alpha_D} + \frac{\sigma_S\,\Omega^{1/2}\,C_G\,\tau_0^{1/2}}{(r+\alpha_D)(r+\alpha_S)}\Big)^2 + \frac{(\sigma_S\,\Omega^{1/2}\,C_G)^2\,N\,\tau_I}{(r+\alpha_D)^2(r+\alpha_S)^2} + \frac{\sigma_L^2}{(r+\alpha_D)^2(r+\alpha_L)^2}.$$

From equation (37), we obtain

$$H(t) = \left(P(t) - \frac{D(t)}{r+\alpha_D} - \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)}\right)\frac{(r+\alpha_D)(r+\alpha_S)}{C_G\,\sigma_S\,\Omega^{1/2}}. \tag{A-76}$$

We also have the following relationship between traders' signals $H_n(t)$ and the and the statistic $\hat{H}_n(t)$, $n = 0, 1, \ldots, N$:

$$\hat{H}_n(t) = H_n(t) + (\alpha_G + \tau - \hat{\alpha}_S - \hat{\tau})\int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_S+\hat{\tau})\,(t-u)}\;H_n(u)\,du. \tag{A-77}$$

Substituting (A-76) and (A-77) into (38) yields (39), where α_1, α_2, and α_3 are defined as

$$\alpha_1 := \left(b\,\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - a\right)\frac{(r+\alpha_D)\,(r+\alpha_S)}{C_G\,\sigma_S\,\Omega^{1/2}}, \tag{A-78}$$

$$\alpha_2 := b\,\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}}\,\frac{(\alpha_S + \tau - \hat{\alpha}_S - \hat{\tau})\,(r+\alpha_D)\,(r+\alpha_S)}{C_G\,\sigma_S\,\Omega^{1/2}}, \tag{A-79}$$

$$\alpha_3 := -b\,\frac{\tau_I^{1/2}\,\hat{\tau}_0^{1/2} - \hat{\tau}_I^{1/2}\,\tau_0^{1/2}}{\tau_I^{1/2}}, \tag{A-80}$$

with a and b defined in equations (A-70) and (A-71).

A.8. Proof of Theorem 3

Using the definitions of $H(t)$ and $\hat{H}(t)$ in equations (32) and (31)—as well as equations (27), (30), and (33)—we can write a continuous 2-vector stochastic process $y_H(t) = [H(t), \hat{H}(t)]'$ satisfying the linear stochastic differential equation

$$dy_H(t) = K_H\, y_H(t)\, dt + C_H\, dZ_H(t), \tag{A-81}$$

where K_H is a 2×2 matrix and C_H is a 2×2 matrix given by

$$K_H = \begin{bmatrix} -\alpha_S - \tau & \tau_0^{1/2}\, \hat{\tau}_0^{1/2} + N\, \hat{\tau}_I^{1/2}\, \tau_I^{1/2} \\ 0 & -\hat{\alpha}_S \end{bmatrix}, \qquad C_H = \begin{bmatrix} \tau_0^{1/2} & N\, \tau_I^{1/2} \\ \hat{\tau}_0^{1/2} & N\, \hat{\tau}_I^{1/2} \end{bmatrix}. \tag{A-82}$$

Using an empirically correct specification, the vector $dZ_H(t) = [dB_0^*(t), d\bar{B}^*(t)]'$ is a 2×1-dimensional Brownian motion, where $dB_0^*(t)$ is a Brownian motion with variance of one defined in equation (A-74), and $d\bar{B}^*(t)$ is a Brownian motion with variance $1/N$ defined in equation (A-73).

Using results about linear continuous-time stochastic processes, we can represent the process $y_H(t) = [H(t), \hat{H}(t)]'$ in an integral form as

$$y_H(s) = \mathrm{e}^{K_H\,(s-t)}\; y_H(t) + \int_{u=t}^{s} \mathrm{e}^{K_H\,(s-u)}\; C_H\, dZ_H(u). \tag{A-83}$$

It can be also shown that the exponential 2×2 matrix $\mathrm{e}^{K_H\, t}$ is given by

$$\mathrm{e}^{K_H\, t} = \begin{bmatrix} \mathrm{e}^{-(\alpha_S+\tau)\, t} & \frac{\tau_0^{1/2}\, \hat{\tau}_0^{1/2} + N\, \hat{\tau}_I^{1/2}\, \tau_I^{1/2}}{\tau + \alpha_S - \hat{\alpha}_S} \left(\mathrm{e}^{-\hat{\alpha}_S\, t} - \mathrm{e}^{-(\alpha_S+\tau)\, t}\right) \\ 0 & \mathrm{e}^{-\hat{\alpha}_S\, t} \end{bmatrix}. \tag{A-84}$$

Plug $\mathrm{e}^{K_H\, t}$ back into equation (A-83) to obtain recursive formulas for the stochastic vector $y_H(s) = [H(s), \hat{H}(s)]'$ as a function of $y_H(t) = [H(t), \hat{H}(t)]'$. Use this to express the cumulative holding period return $R(t, t+T)$ as a linear function of $H(t)$ and $\hat{H}(t)$:

$$R(t, t+T) = \zeta_2(T)\, \hat{H}(t) - \zeta_1(T)\, H(t) + \frac{(\alpha_L - \hat{\alpha}_L)(1 - \mathrm{e}^{-\hat{\alpha}_L T})}{(r + \alpha_D)(r + \alpha_L)\hat{\alpha}_L} G_L(t) + \bar{B}(t, t+T), \tag{A-85}$$

where

(A-86)
$$\bar{B}(t,t+T) := \int_{s=t}^{t+T}\int_{u=s}^{t+T}[-a,b]\,\mathrm{e}^{K_H(u-s)}\,C_H\,du\,dZ_H(s) + \int_{s=t}^{t+T}\frac{\sigma_L}{(r+\alpha_D)(r+\alpha_L)}dB_L(s)$$
$$+\int_{s=t}^{t+T} d\hat{B}_r(s) + \frac{(\alpha_L-\hat{\alpha}_L)\sigma_L}{(r+\alpha_D)(r+\alpha_L)}\int_{s=t}^{t+T}\int_{u=s}^{t+T}\mathrm{e}^{-\hat{\alpha}_L(u-s)}\;du\,dB_L(s),$$

(A-87)
$$\zeta_1(T) := \frac{a}{\alpha_S+\tau}\left(1-\mathrm{e}^{-(\alpha_S+\tau)T}\right),$$

(A-88)
$$\zeta_2(T) := b\frac{1-\mathrm{e}^{-\hat{\alpha}_S T}}{\hat{\alpha}_S} - a\frac{\tau_0^{1/2}\hat{\tau}_0^{1/2}+N\hat{\tau}_I^{1/2}\tau_I^{1/2}}{\hat{\alpha}_S\,(\alpha_S+\tau)}\left(1+\frac{\hat{\alpha}_S\,\mathrm{e}^{-(\alpha_S+\tau)T}-(\alpha_S+\tau)\,\mathrm{e}^{-\hat{\alpha}_S T}}{\tau+\alpha_S-\hat{\alpha}_S}\right).$$

The constants a and b are as defined in equations (A-70) and (A-71). It can be shown that $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$. Since both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$, dividends $D(t)$, and long-term dividend growth rate $G_L(t)$, the cumulative holding-period return $R(t,t+T)$ can be expressed as in equation (42), where

(A-89)
$$\beta_1(T) := \left(\zeta_2(T)\,\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - \zeta_1(T)\right)\,\frac{(r+\alpha_D)\,(r+\alpha_S)}{C_G\,\sigma_S\,\Omega^{1/2}},$$

(A-90)
$$\beta_2(T) := \zeta_2(T)\,\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}}\,\frac{(\alpha_S+\tau-\hat{\alpha}_S-\hat{\tau})\,(r+\alpha_D)\,(r+\alpha_S)}{C_G\,\sigma_S\,\Omega^{1/2}},$$

(A-91)
$$\beta_3(T) := \zeta_2(T)\,\frac{\hat{\tau}_I^{1/2}\,\tau_0^{1/2}-\tau_I^{1/2}\,\hat{\tau}_0^{1/2}}{\tau_I^{1/2}}.$$

A.9. Proof of Proposition 3

When the traders use an empirically correct value for total precision of information flow and the parameters describing the short-term growth rate, the expected holding-period excess return can be written

(A-92)
$$\hat{\mathrm{E}}_t\{R(t,t+T)\} = \left(\hat\tau_I^{1/2}\zeta_2(T) - \tau_I^{1/2}\zeta_1(T)\right)\frac{(r+\alpha_D)(r+\alpha_S)}{C_G\sigma_S\Omega^{1/2}\tau_I^{1/2}}\left(P(t) - \frac{D(t)}{r+\alpha_D} - \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)}\right)$$
$$-\frac{(\hat\alpha_L-\alpha_L)(1-\mathrm{e}^{-\hat\alpha_L T})}{(r+\alpha_D)(r+\alpha_L)\hat\alpha_L}G_L(t) - \left(\hat\tau_I^{1/2}/\tau_I^{1/2}-1\right)\zeta_2(T)\,\tau_0^{1/2}H_0(t).$$

Direct computation shows that the coefficient of $P(t) - \frac{D(t)}{r+\alpha_D} - \frac{G_L(t)}{(r+\alpha_D)(r+\alpha_L)}$ in Proposition 3 is positive and monotonically increases in the horizon T. The coefficient of $H_0(t)$ in Proposition 3 is negative. In addition, it can be shown that the coefficient of the first term, $\zeta_2(T)\hat\tau_I^{1/2} - \zeta_1(T)\tau_I^{1/2} > 0$, can be decomposed to two terms:

(A-93)
$$\frac{\sigma_S\Omega^{1/2}}{r+\alpha_D}(1-C_G)\tau_I^{1/2}\frac{1-\mathrm{e}^{-\alpha_S T}}{\alpha_S} + \frac{\sigma_S\Omega^{1/2}}{(r+\alpha_D)\alpha_S(r+\alpha_S)}\left(\hat\tau_I^{1/2}-\tau_I^{1/2}\right)$$
$$\cdot\left(\left(1-\mathrm{e}^{-\alpha_S T}\right)\left(r+\alpha_S-\frac{C_G r\tau_0}{\alpha_S+\tau}\right) + \frac{C_G(\alpha_S+r+\tau)\alpha_S\tau_0}{(\alpha_S+\tau)\tau}\left(\mathrm{e}^{-\alpha_S T}-\mathrm{e}^{-(\alpha_S+\tau)T}\right)\right).$$

The first term with $1-C_G>0$ results from the price dampening effect of the Keynesian beauty contest, and the second term with $\hat\tau_I^{1/2}-\tau_I^{1/2}>0$ results from the price dampening effect of $C_J<1$.

A.10. Covariance and Correlation of $R(t-T_l,t)$ and $R(t,t+T_f)$

We first derive the steady-state unconditional variance-covariance matrix of $H(t)$ and $\hat H(t)$. Define the steady-state unconditional variance-covariance matrix of $H(t)$ and $\hat H(t)$ as $Q = \big[[q_{11},q_{12}],[q_{12},q_{22}]\big]$. In the steady state, we have

(A-94)
$$K_h\,Q + Q\,K_h' + C_h\,C_h' = 0.$$

It can be shown that

(A-95)
$$q_{11} = \frac{\tau_0+N\,\tau_I}{2(\alpha_S+\tau)} + \frac{(2\hat\alpha_S+\hat\tau)\,(\tau_0^{1/2}\hat\tau_0^{1/2}+N\,\hat\tau_I^{1/2}\,\tau_I^{1/2})^2}{2\hat\alpha_S\,(\alpha_S+\hat\alpha_S+\tau)\,(\alpha_S+\tau)},$$

(A-96)
$$q_{12} = \frac{(2\hat\alpha_S+\hat\tau)\,(\tau_0^{1/2}\,\hat\tau_0^{1/2}+N\,\hat\tau_I^{1/2}\,\tau_I^{1/2})}{2\hat\alpha_S\,(\alpha_S+\tau+\hat\alpha_S)},\qquad q_{22} = \frac{\hat\tau}{2\hat\alpha_S}.$$

We now calculate the covariance of $R(t-T_l,t)$ and $R(t,t+T_f)$. Since the unconditional means of $R(t-T_l,t)$ and $R(t,t+T_f)$ are zero, equations (A-83) and (A-85) yield

(A-97)

$$\begin{aligned}
&\mathrm{Cov}\{R(t-T_l,t),R(t,t+T_f)\} = \hat{\mathrm{E}}\{R(t-T_l,t)\,R(t,t+T_f)\}\\
&\quad = \hat{\mathrm{E}}\Big\{\Big(-\zeta_1(T_l)\,H(t-T_l)+\zeta_2(T_l)\,\hat{H}(t-T_l)+\frac{(\alpha_L-\hat{\alpha}_L)(1-\mathrm{e}^{-\hat{\alpha}_L T_l})}{(r+\alpha_D)(r+\alpha_L)\hat{\alpha}_L}G_L(t-T_l)+\bar{B}(t-T_l,t)\Big)\\
&\quad \times\ \Big(-\zeta_1(T_f)\,H(t)+\zeta_2(T_f)\,\hat{H}(t)+\frac{(\alpha_L-\hat{\alpha}_L)(1-\mathrm{e}^{-\hat{\alpha}_L T_f})}{(r+\alpha_D)(r+\alpha_L)\hat{\alpha}_L}G_L(t)\Big)\Big\},
\end{aligned}$$

where $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ are as defined in equations (A-87) and (A-88). Define

$$\eta := \frac{\tau_0^{1/2}\,\hat{\tau}_0^{1/2}+N\,\hat{\tau}_I^{1/2}\,\tau_I^{1/2}}{\tau+\alpha_S-\hat{\alpha}_S},\qquad m_1 := -\frac{a(\hat{\tau}_0^{1/2}\eta-\tau_0^{1/2})^2}{\alpha_S+\tau}\zeta_1(T_f), \tag{A-98}$$

$$m_3 := (\hat{\tau}_0^{1/2}\,\eta-\tau_0^{1/2})\hat{\tau}_0^{1/2}\left(\frac{-a\,(\zeta_2(T_f)-\eta\,\zeta_1(T_f))}{\alpha_S+\tau}+\frac{-\zeta_1(T_f)(-a\,\eta+b)}{\hat{\alpha}_S}\right), \tag{A-99}$$

(A-100)

$$m_4 := \zeta_1(T_f)(\tau_0^{1/2}-\hat{\tau}_0^{1/2}\eta)\left(\frac{\hat{\tau}_0^{1/2}(-a\eta+b)}{\hat{\alpha}_S}+\frac{\sigma_S\Omega^{1/2}C_G\tau_0^{1/2}+\sigma_D(r+\alpha_S)}{(r+\alpha_D)\,(r+\alpha_S)}+\frac{a(\hat{\tau}_0^{1/2}\eta-\tau_0^{1/2})}{\alpha_S+\tau}\right),$$

$$m_2 := -\frac{\hat{\tau}_0\,(-a\eta+b)}{\hat{\alpha}_S}(\zeta_2(T_f)-\eta\,\zeta_1(T_f)),\quad m_5 := -\frac{(\zeta_2(T_f)-\eta\,\zeta_1(T_f))\,\hat{\tau}_0^{1/2}}{\zeta_1(T_f)\,(\tau_0^{1/2}-\hat{\tau}_0^{1/2}\,\eta)}\,m_4, \tag{A-101}$$

$$n_1 := -\zeta_1(T_f)N\frac{a(\hat{\tau}_I^{1/2}\eta-\tau_I^{1/2})^2}{\alpha_S+\tau},\quad n_2 := -(-\zeta_1(T_f)\eta+\zeta_2(T_f))\frac{N\hat{\tau}_I(-a\eta+b)}{\hat{\alpha}_S}, \tag{A-102}$$

$$n_3 := N\hat{\tau}_I^{1/2}(\hat{\tau}_I^{1/2}\eta-\tau_I^{1/2})\left(\frac{-a}{\alpha_S+\tau}(-\zeta_1(T_f)\eta+\zeta_2(T_f))-\zeta_1(T_f)\frac{-a\eta+b}{\hat{\alpha}_S}\right), \tag{A-103}$$

(A-104)

$$n_4 := -N\zeta_1(T_f)(\tau_I^{1/2}-\hat{\tau}_I^{1/2}\eta)\left(\frac{-a(\tau_I^{1/2}-\hat{\tau}_I^{1/2}\eta)}{\alpha_S+\tau}+\frac{\hat{\tau}_I^{1/2}(-a\eta+b)}{\hat{\alpha}_S}+\frac{\sigma_S\Omega^{1/2}C_G\tau_I^{1/2}}{(r+\alpha_D)(r+\alpha_S)}\right),$$

$$n_5 := -\frac{(\zeta_2(T_f)-\eta\,\zeta_1(T_f))\,\hat{\tau}_I^{1/2}}{\zeta_1(T_f)\,(\tau_I^{1/2}-\hat{\tau}_I^{1/2}\,\eta)}\,n_4. \tag{A-105}$$

Then direct calculations show that $\mathrm{Cov}\{R(t-T_l,t),R(t,t+T_f)\}$ is a function of T_l and T_f given by

(A-106)

$$\begin{aligned}
&\mathrm{Cov}\{R(t-T_l,t),R(t,t+T_f)\} = \\
&\quad -\zeta_1(T_f)\,(-\zeta_1(T_l)\,(q_{11}-q_{12}\,\eta)+\zeta_2(T_l)\,(q_{12}-q_{22}\,\eta))\ \mathrm{e}^{-(\alpha_S+\tau)T_l} \\
&\quad +(\zeta_2(T_l)\,q_{22}-\zeta_1(T_l)\,q_{12})\,(-\zeta_1(T_f)\,\eta+\zeta_2(T_f))\ \mathrm{e}^{-\hat{\alpha}_S T_l}+\frac{m_1+n_1}{2(\alpha_S+\tau)}(1-\mathrm{e}^{-2(\alpha_S+\tau)T_l}) \\
&\quad +\frac{m_2+n_2}{2\hat{\alpha}_S}(1-\mathrm{e}^{-2\hat{\alpha}_S T_l})+\frac{m_3+n_3}{\alpha_S+\tau+\hat{\alpha}_S}(1-\mathrm{e}^{-(\alpha_S+\tau+\hat{\alpha}_S)T_l})+\frac{m_4+n_4}{\alpha_S+\tau}(1-\mathrm{e}^{-(\alpha_S+\tau)T_l}) \\
&\quad +\frac{m_5+n_5}{\hat{\alpha}_S}(1-\mathrm{e}^{-\hat{\alpha}_S T_l})+\frac{\sigma_L^2(\alpha_L^2-\hat{\alpha}_L^2)(1-\mathrm{e}^{-\hat{\alpha}_L T_l})(1-\mathrm{e}^{-\hat{\alpha}_L T_f})}{2\hat{\alpha}_L^3(r+\alpha_D)^2(r+\alpha_L)^2}.
\end{aligned}$$

To calculate the correlation coefficients of $R(t-T_l,t)$ and $R(t,t+T_f)$, we now calculate variances $\mathrm{Var}\{R(t-T_l,t)\}$ and $\mathrm{Var}\{R(t,t+T_f)\}$. Define

(A-107) $$k_1 := \frac{a^2(\hat{\tau}_0^{1/2}\eta-\tau_0^{1/2})^2+a^2N(\hat{\tau}_I^{1/2}\eta-\tau_I^{1/2})^2}{(\alpha_S+\tau)^2},\qquad k_2 := \frac{(-a\eta+b)^2(\hat{\tau}_0+N\hat{\tau}_I)}{\hat{\alpha}_S^2},$$

(A-108) $$k_3 := \frac{2a(-a\eta+b)(\hat{\tau}_0^{1/2}(\hat{\tau}_0^{1/2}\eta-\tau_0^{1/2})+N\hat{\tau}_I^{1/2}(\hat{\tau}_I^{1/2}\eta-\tau_I^{1/2}))}{(\alpha_S+\tau)\hat{\alpha}_S},$$

(A-109) $$c_{k1} := \frac{\hat{\tau}_0^{1/2}(-a\eta+b)}{\hat{\alpha}_S}+\frac{\sigma_S\Omega^{1/2}C_G\tau_0^{1/2}+\sigma_D(r+\alpha_S)}{(r+\alpha_D)\,(r+\alpha_S)}+\frac{a(\hat{\tau}_0^{1/2}\eta-\tau_0^{1/2})}{\alpha_S+\tau},$$

(A-110) $$c_{k2} := N\,\left(\frac{-a(\tau_I^{1/2}-\hat{\tau}_I^{1/2}\eta)}{\alpha_S+\tau}+\frac{\hat{\tau}_I^{1/2}(-a\eta+b)}{\hat{\alpha}_S}+\frac{\sigma_S\Omega^{1/2}C_G\tau_I^{1/2}}{(r+\alpha_D)(r+\alpha_S)}\right),$$

(A-111) $$k_4 := -2a\,\frac{c_{k1}\,(\hat{\tau}_0^{1/2}\eta-\tau_0^{1/2})+c_{k2}\,(\tau_I^{1/2}\eta-\tau_I^{1/2})}{\alpha_S+\tau},$$

(A-112) $$k_5 := -\frac{2(-a\eta+b)(c_{k1}\,\hat{\tau}_0^{1/2}+c_{k2}\,\hat{\tau}_I^{1/2})}{\hat{\alpha}_S},\qquad k_6 := c_{k1}^2+\frac{1}{N}c_{k2}^2.$$

Then direct calculations show that $\mathrm{Var}\{R(t-T_l,t)\}$ and $\mathrm{Var}\{R(t,t+T_f)\}$ are

(A-113)

$$\mathrm{Var}\{R(t-T_l,t)\} = q_{11}\,\zeta_1^2(T_l) + q_{22}\,\zeta_2^2(T_l) - 2q_{12}\zeta_1(T_l)\zeta_2(T_l) + \frac{k_1}{2(\alpha_S+\tau)}(1-\mathrm{e}^{-2(\alpha_S+\tau)T_l}) + \frac{k_2}{2\hat{\alpha}_S}(1-\mathrm{e}^{-2\hat{\alpha}_S T_l}) + \frac{k_3}{\alpha_S+\tau+\hat{\alpha}_S}(1-\mathrm{e}^{-(\alpha_S+\tau+\hat{\alpha}_S)T_l}) + \frac{k_4}{\alpha_S+\tau}(1-\mathrm{e}^{-(\alpha_S+\tau)T_l}) + \frac{k_5}{\hat{\alpha}_S}(1-\mathrm{e}^{-\hat{\alpha}_S T_l}) + k_6 T_l + \frac{\sigma_L^2}{\hat{\alpha}_L^3(r+\alpha_D)^2(r+\alpha_L)^2}\left(\alpha_L^2\hat{\alpha}_L T_l - (\alpha_L^2-\hat{\alpha}_L^2)(1-\mathrm{e}^{-\hat{\alpha}_L T_l})\right),$$

(A-114)

$$\mathrm{Var}\{R(t,t+T_f)\} = q_{11}\,\zeta_1^2(T_f) + q_{22}\,\zeta_2^2(T_f) - 2q_{12}\zeta_1(T_f)\zeta_2(T_f) + \frac{k_1}{2(\alpha_S+\tau)}(1-\mathrm{e}^{-2(\alpha_S+\tau)T_f}) + \frac{k_2}{2\hat{\alpha}_S}(1-\mathrm{e}^{-2\hat{\alpha}_S T_f}) + \frac{k_3}{\alpha_S+\tau+\hat{\alpha}_S}(1-\mathrm{e}^{-(\alpha_S+\tau+\hat{\alpha}_S)T_f}) + \frac{k_4}{\alpha_S+\tau}(1-\mathrm{e}^{-(\alpha_S+\tau)T_f}) + \frac{k_5}{\hat{\alpha}_S}(1-\mathrm{e}^{-\hat{\alpha}_S T_f}) + k_6 T_f + \frac{\sigma_L^2}{\hat{\alpha}_L^3(r+\alpha_D)^2(r+\alpha_L)^2}\left(\alpha_L^2\hat{\alpha}_L T_f - (\alpha_L^2-\hat{\alpha}_L^2)(1-\mathrm{e}^{-\hat{\alpha}_L T_f})\right).$$

Then, using equations (A-106), (A-113), and (A-114), we get the correlation coefficient of $R(t-T_l,t)$ and $R(t,t+T_f)$. From equations (A-106) and (A-113), we obtain the regression coefficient of regressing the cumulative return $R(t,t+T_f)$ on $R(t-T_l,t)$ as

(A-115)
$$b_{T_f,T_l} = \frac{\mathrm{Cov}\{R(t-T_l,t),R(t,t+T_f)\}}{\mathrm{Var}\{R(t-T_l,t)\}}.$$

Consider the case when traders are correct on average, so that the correct precision is $\hat{\tau}_I = (\tau_H + (N-1)\tau_L)/N$ to each signal. Since $\hat{\tau}_0 = \tau_0$ and $\hat{\Omega} = \Omega$ hold in this case, the two signals coincide, yielding $\hat{H}_n(t) = H_n(t)$. It can be shown that

(A-116)

$$\mathrm{Cov}\{R(t-T,t),R(t,t+T)\} = \hat{\mathrm{E}}\{R(t-T,t)\,R(t,t+T)\} = \frac{\sigma_S^2\Omega}{(r+\alpha_D)^2}\frac{(1-\mathrm{e}^{-\alpha_S T})^2}{\alpha_S^2}\frac{\tau - C_G(\tau_0+N\hat{\tau}_I^{1/2}\tau_I^{1/2})}{\tau}\left(1+\frac{\tau}{2\alpha_S}+\frac{C_G(\alpha_S-r)(\tau_0+N\hat{\tau}_I^{1/2}\tau_I^{1/2})}{2\alpha_S(r+\alpha_S)}\right) - \frac{\sigma_S^2\Omega N(\hat{\tau}_I^{1/2}-\tau_I^{1/2})C_G}{(r+\alpha_D)^2}\frac{(1-\mathrm{e}^{-(\alpha_S+\tau)T})^2}{(\alpha_S+\tau)^2}\frac{r+\alpha_S+\tau}{(r+\alpha_S)\tau}\left(\frac{C_G\tau_0(\hat{\tau}_I^{1/2}-\tau_I^{1/2})}{2(r+\alpha_S)}\frac{\alpha_S+\tau-r}{\alpha_S+\tau}+\hat{\tau}_I^{1/2}\right) + \frac{\sigma_L^2(\alpha_L^2-\hat{\alpha}_L^2)(1-\mathrm{e}^{-\hat{\alpha}_L T})^2}{2\hat{\alpha}_L^3(r+\alpha_D)^2(r+\alpha_L)^2}.$$

The dampening effect ($C_G < 1$) leads to momentum (positive autocorrelation) in returns.

If $\alpha_L = \hat{\alpha}_L$, for the limiting case studied in Section 1.3 with $\tau_L = 0$, $\tau_0 \to 0$, and $N \to \infty$, the dampening effect is substantial; it can be shown that

$$\text{(A-117)}\quad \mathrm{Cov}\{R(t-T,t), R(t,t+T)\} = \frac{(1-\mathrm{e}^{-\alpha_S T})^2}{\alpha_S^2}\left(1+\frac{\tau}{2\alpha_S}\right) - \frac{(1-\mathrm{e}^{-(\alpha_S+\tau)T})^2}{(\alpha_S+\tau)^2} > 0.$$

For general cases, from (A-116), the autocovariance of holding-period returns tends to be positive when τ is large relative to τ_0 and α_S. In extensive numerical analysis of equation (A-116) over a large range of parameters, we always find a positive autocovariance.

B. Online Appendix

B.1. An Analogous One-Period Competitive Model

A risky asset with random liquidation value $v \sim N(0, 1/\tau_v)$ is traded for a safe numeraire asset. Each of N traders $n = 1, \ldots, N$ is endowed with S_n shares of a zero-net-supply risky asset, implying $\sum_{n=1}^{N} S_n = 0$. Traders observe signals about the normalized liquidation value $\tau_v^{1/2} v$. All traders observe a public signal $i_0 := \tau_0^{1/2} \, (\tau_v^{1/2} v) + e_0$ with $e_0 \sim N(0, 1)$. Each trader n observes a private signal $i_n := \tau_n^{1/2} \, (\tau_v^{1/2} v) + e_n$ with $e_n \sim N(0, 1)$. The asset payoff v, the public signal error e_0, and N private signal errors $e_1, \ldots, e_N$ are independently distributed.

Traders agree about the precision of the public signal τ_0 and agree to disagree about the precisions of private signals τ_n. Each trader is "relatively overconfident," believing his own signal has a high precision $\tau_n = \tau_H$ and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Let $\mathrm{E}^n\{\ldots\}$ and $\mathrm{Var}^n\{\ldots\}$ denote trader n's expectation and variance operators conditional on all signals $i_0, i_1, \ldots, i_N$. Define "total precision" τ by

$$\tau := \left(\mathrm{Var}^n\{v\}\right)^{-1} = \tau_v \, \left(1 + \tau_0 + \tau_H + (N-1)\, \tau_L\right). \tag{B-1}$$

The projection theorem for jointly normally distributed random variables implies

$$\mathrm{E}^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \, \left(\tau_0^{1/2} \, i_0 + \tau_H^{1/2} \, i_n + (N-1)\, \tau_L^{1/2} \, i_{-n}\right). \tag{B-2}$$

Each trader submits a demand schedule $X_n(p) := X_n(i_0, i_n, S_n, p)$ to a single-price auction. An auctioneer calculates the market-clearing price $p := p[X_1, \ldots, X_N]$.

Trader n's terminal wealth is

$$W_n := v \, (S_n + X_n(p)) - p \, X_n(p). \tag{B-3}$$

Each trader maximizes the same expected exponential utility function of wealth $\mathrm{E}^n\{-\,\mathrm{e}^{-A\,W_n}\}$ using his own beliefs about τ_H and τ_L to calculate the expectation.

Trader n maximizes his expected utility, or equivalently he maximizes $\mathrm{E}^n\{W_n\} - \frac{1}{2} A \, \mathrm{Var}^n\{W_n\}$.

He chooses the quantity to trade x_n that solves the maximization problem
(B-4)

$$\max_{x_n} \left\{ \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2}\, i_0 + \tau_H^{1/2}\, i_n + (N-1)\tau_L^{1/2}\, i_{-n} \right) (S_n + x_n) - p\, x_n - \frac{A}{2\tau}(S_n + x_n)^2 \right\}.$$

The first-order condition with respect to x_n yields

$$x_n^* = \tfrac{1}{A} \left(\tau_v^{1/2} \left(\tau_0^{1/2}\, i_0 + \tau_H^{1/2}\, i_n + (N-1)\tau_L^{1/2}\, i_{-n} \right) - p\, \tau \right) - S_n. \tag{B-5}$$

The market-clearing condition $\sum_{n=1}^N x_n^* = 0$ implies

$$p^* = \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2}\, i_0 + \tfrac{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}{N} \sum_{n=1}^{N} i_n \right). \tag{B-6}$$

Substituting (B-6) into (B-5) yields

$$x_n^* = \tfrac{1}{A}\left(1 - \tfrac{1}{N}\right)\tau_v^{1/2}(\tau_H^{1/2} - \tau_L^{1/2})(i_n - i_{-n}) - S_n. \tag{B-7}$$

Thus, each trader trades on the difference between his signal i_n and the average of all N signals. Equation (B-6) implies that the equilibrium price is a weighted average of traders' valuations about the fundamental value of the asset with weights summing to one.

Define the constant C_J as the ratio of the average of the square roots to the square root of the average of precisions:

$$C_J := \left(\tfrac{1}{N}\, \tau_H^{1/2} + \tfrac{N-1}{N}\, \tau_L^{1/2} \right) \left(\tfrac{1}{N}\, \tau_H + \tfrac{N-1}{N}\, \tau_L \right)^{-1/2}. \tag{B-8}$$

We can rewrite the price as

$$p^* = \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2}\, i_0 + C_J \left(\tfrac{1}{N}\tau_H + \tfrac{N-1}{N}\tau_L \right)^{1/2} \sum_{n=1}^{N} i_n \right). \tag{B-9}$$

When traders are relatively overconfidence ($\tau_H > \tau_L$), Jensen's inequality implies that $C_J < 1$. Therefore, the "static dampening effect" of $C_J < 1$ shows up in our one-period model.

B.2. The Expected Holding-Period Returns for Cases with Absolute Overconfidence

We now look at the general case when investors use an empirically incorrect value for the total precision of the signals and parameter values of the short-term growth rate. The results when use correct values of the parameter values can be obtained by setting $\hat{\alpha}_S = \alpha_S$ and $\hat{\sigma}_S = \sigma_S$. We assume $\alpha_S + \tau > \hat{\alpha}_S$.[1] Define

(B-10)
$$\nu_1 := \frac{\tau_0^{1/2}\hat{\tau}_0^{1/2} + N\,\hat{\tau}_I^{1/2}\,\tau_I^{1/2}}{\alpha_S - \hat{\alpha}_S + \tau}, \qquad \nu_3 := \frac{C_G(r+\hat{\alpha}_S)\,(2\hat{\alpha}_S+\hat{\tau})^{1/2}}{(r+\alpha_S)\,(2\alpha_S+\tau)^{1/2}}\nu_1,$$

(B-11)
$$\nu_2 = \nu_1 + \frac{(r+\alpha_S)\,\hat{\Omega}^{1/2}\,(\hat{\sigma}_S - \nu_3\sigma_S)}{C_G\,(r+\alpha_S+\tau)\,\Omega^{1/2}\,\sigma_S},$$

and

(B-12)
$$T_2 := \frac{1}{\alpha_S+\tau-\hat{\alpha}_S}\ln\left(\frac{C_G\,\sigma_S\,(r+\alpha_S+\tau)\,\Omega^{1/2}\left(H(t)-\nu_1\hat{H}(t)\right)}{(r+\alpha_S)\,\hat{\Omega}^{1/2}\,(\hat{\sigma}_S-\nu_3\sigma_S)\,\hat{H}(t)}\right),$$

it can be shown that $T_2 > 0$ if and only if $H(t) > \nu_2\hat{H}(t)$. We also have

(B-13)
$$\frac{d\hat{\mathrm{E}}_t\{R(t,t+T)\}}{dT} = \frac{\mathrm{e}^{-(\alpha_S+\tau)T}}{(r+\alpha_D)\,(r+\alpha_S)}\Big(-C_G\,\sigma_S\,(r+\alpha_S+\tau)\,\Omega^{1/2}\,(H(t)-\nu_1\hat{H}(t)) + (r+\alpha_S)\,\hat{\Omega}^{1/2}\,(\hat{\sigma}_S-\nu_3\sigma_S)\,\hat{H}(t)\;\mathrm{e}^{(\alpha_S+\tau-\hat{\alpha}_S)T}\Big).$$

We assume $\hat{\alpha}_L = \alpha_L$ to examine the return patterns when traders have empirically incorrect beliefs about the total precision of the signals and parameter values of the short-term growth rate. The following proposition shows that there are only four possible patterns of the expected holding period return: only momentum, only mean-reversion, first mean-reversion and then momentum, first momentum and then mean-reversion.

PROPOSITION 1: *Assume $\hat{\alpha}_L = \alpha_L$. When investors use empirically incorrect beliefs about both the total precision of the signals and parameter values of the model, we obtain four cases:*

1) *If $H(t) \le \nu_1\hat{H}(t)$ and $\hat{H}(\hat{\sigma}_S - \nu_3\sigma_S) \ge 0$, then $\hat{\mathrm{E}}_t\{R(t,t+T)\}$ monotonically increases in T.*

[1] The case with $\alpha_S + \tau < \hat{\alpha}_S$ is similar.

2) If $H(t) \leq \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_S - \nu_3 \sigma_S) < 0$, then $\hat{\mathrm{E}}_t\{R(t, t+T)]\}$ increases in T for $T < T_2$ and decreases in T for $T > T_2$.

3) If $H(t) > \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_S - \nu_3 \sigma_S) \leq 0$, then $\hat{\mathrm{E}}_t\{R(t, t+T)\}$ monotonically decreases in T.

4) If $H(t) > \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_S - \nu_3 \sigma_S) > 0$, then $\hat{\mathrm{E}}_t\{R(t, t+T)\}$ decreases in T for $T < T_2$ and increases in T for $T > T_2$.

Proposition 1 implies that the expected holding period returns $\hat{\mathrm{E}}_t\{R(t, t+T)\}$ can be monotonically increasing or decreasing over time T, or it might be increasing first then decreasing over time T, or it might be decreasing first then increasing over time T. Whether $\hat{\mathrm{E}}_t\{R(t, t+T)\}$ increases or decreases in time T depends on the relative magnitude of the current signals of $H(t)$ and $\hat{H}(t)$. It also depends differences between empirically correct and trader parameters for mean-reversion rate and volatility of the dividend growth, α_S and σ_S. $\hat{\mathrm{E}}_t\{R(t, t+T)\}$ converges to a constant when $T \to \infty$. As illustrated in Proposition 1, our model may generate short-run momentum and long-run reversal in the term structure of the returns as observed in the data when the traders use incorrect values of the precisions, mean-reverting rate, and the volatility of the growth rate of the dividend ($\hat{\tau} < \tau$, $\hat{\sigma}_S \neq \sigma_S$, and $\hat{\alpha}_S \neq \alpha_S$.

Assume investors are absolutely overconfident in the sense that $\hat{\tau} < \tau$, and assume the investors use empirically correct parameters $\hat{\alpha}_S = \alpha_S$ and $\hat{\sigma}_S = \sigma_S$. For this case, it can be shown that

$$\nu_3 = \frac{C_G(2\alpha_S + \hat{\tau})^{1/2}}{(2\alpha_S + \tau)^{1/2}} \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N\, \hat{\tau}_I^{1/2}\, \tau_I^{1/2}}{\tau} < 1, \tag{B-14}$$

since $\hat{\Omega} > \Omega$, $C_G < 1$, and $\tau_0^{1/2} \hat{\tau}_0^{1/2} + N\, \hat{\tau}_I^{1/2}\, \tau_I^{1/2} < \tau$. This implies that we will obtain cases (1) and (4) in proposition 1 for positive signals $\hat{H}(t)$; specifically, we obtain (1) only momentum or (4) short-run reversal and long-run momentum.

From proposition 1, if $H(t)/\nu_1 \leq \hat{H}(t) < H(t)/\nu_2$, and

$$\sigma_S > \frac{(r + \alpha_S)\,(2\alpha_S + \tau)^{1/2}}{C_G(r + \hat{\alpha}_S)\,(2\hat{\alpha}_S + \hat{\tau})^{1/2}} \frac{\alpha_S - \hat{\alpha}_S + \tau}{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N\, \hat{\tau}_I^{1/2}\, \tau_I^{1/2}} \hat{\sigma}_S, \tag{B-15}$$

where ν_1 and ν_2 are defined in (B-10) and (B-11), equation (B-15) implies that we tend to have short-run momentum and long-run reversal if traders believe that the growth rate is highly volatile (high values of σ_S) or highly persistent (low values of α_S).

Now, we assume $\hat{\alpha}_L = \alpha_L$ and provide examples for two specific combinations of different parameter values (cases I and II). When investors have correct beliefs about the total precision of the information flow ($\hat{\tau} = \tau$) and other parameters of the model ($\hat{\alpha}_S = \alpha_S$ and $\hat{\sigma}_S = \sigma_S$), we tend to have momentum in returns due to the price dampening effect, as shown in Section 2.2 of our paper.

Case I. Figure 1 illustrates the correlation of the cumulative return for the case when traders use correct short-term growth parameters ($\hat{\alpha}_S = \alpha_S$ and $\hat{\sigma}_S = \sigma_S$) but disagree about the total precision of the information flow. Traders are absolutely overconfident ($\tau > \hat{\tau}$).[2]

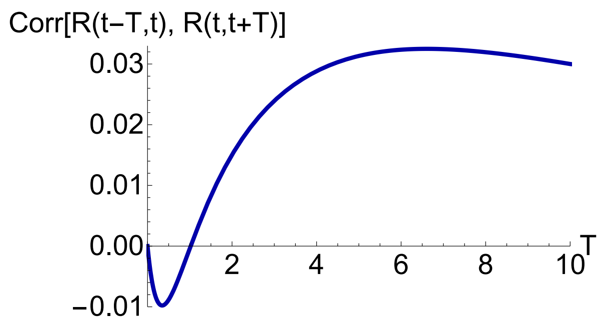

FIGURE 1. THE CORRELATION $\mathrm{Corr}\{R(t-T,t), R(t,t+T)\}$ AGAINST T, WITH $\tau > \hat{\tau}$, $\hat{\alpha}_S = \alpha_S$, AND $\hat{\sigma}_S = \sigma_S$.

Proposition 1 in Appendix B.2 analytically proves that there are only two patterns of the expected holding period return in this case: (1) only momentum, or (2) short-run reversal followed by long-run momentum. Proposition 1 gives specific conditions for each pattern to occur.

As illustrated in Figure 1, the momentum effect dominates return dynamics for most situations. Strong momentum makes most of the correlations positive. Some correlations have negative values at very short horizons.

Case II. Figure 2 illustrates a more general case when traders have incorrect beliefs about both the total precision of the information flow and the parameters of the model.[3] From Proposition 1 in Appendix B.2, the implied return patterns are empirically realistic in the sense that the return exhibits momentum in the short run and mean-reversion in the long run. There is mean reversion in returns if traders believe that the growth rate is more

[2]The parameters are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_S = 0.2$, $\sigma_D = 0.5$, $\sigma_S = 0.1$, $\tau_L = 0.019$, $\tau_H = 0.1$, and $N = 100$, $\hat{\tau} = 1.6$, and $\tau = 2.02$.

[3]We assume $\tau = 2.02 > \hat{\tau} = 1.6$, $\hat{\alpha}_S = 1 > \alpha_S = 0.2$, and $\hat{\sigma}_S = 0.5 > \sigma_S = 0.1$.

persistent than it actually is. As our examples show, the expected holding-period return exhibits different patterns depending on the parameter values.

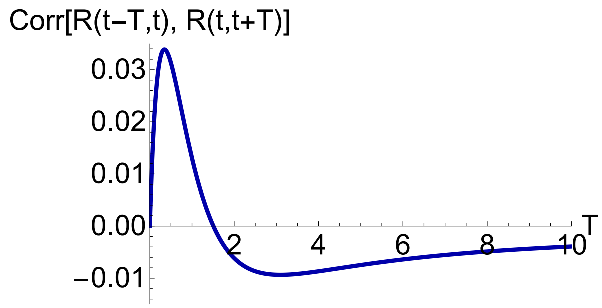

FIGURE 2. THE CORRELATION $\mathrm{Corr}\{R(t-T,t), R(t,t+T)\}$ AGAINST T, WITH $\tau > \hat{\tau}$, $\hat{\alpha}_S \neq \alpha_S$, AND $\hat{\sigma}_S \neq \sigma_S$.

B.3. Predictive Power of Proxies for Momentum and Value Effects Using Simulated Data

We examine the predictive power of past six month return $Return_{t-6m,t,i}$ (a proxy for momentum effect) and the difference between current price and the present value of dividend $Value_{t,i} := P_i(t) - D_i(t)/(r + \alpha_D)$ (a proxy for value effect) for simulation i in predicting next month return $Return_{t,t+1m,i}$ using data from 2000 simulations of monthly excess return over 40 years based on the instantaneous excess return given in equation (38) and the calibrated parameter values in Table 2. Our purpose is not to conduct an extensive analysis on the predictive power of certain firm characteristics in predicting future returns. Instead, the analysis in the subsection illustrates the opposing effects of momentum and value in predicting returns.

We then run panel regressions on simulated data. Table 1 reports the regression coefficients and R^2 of the following models:[4]

$$\text{Model } (F): \quad Return_{t,t+1m,i} = a_0 + a_1 Return_{t-6m,t,i} + e_{t+1m,i},$$

$$\text{Model } (G): \quad Return_{t,t+1m,i} = b_0 + b_1 Return_{t-6m,t,i} + b_2 Value_{t,i} + e_{t+1m,i},$$

$$\begin{aligned}\text{Model } (H): \quad Return_{t,t+1m,i} = {} & c_0 + c_1 Return_{t-6m,t,i} + c_2 Value_{t,i} + c_3 Return_{t-12m,t-7m,i} \\ & + c_4 Return_{t-36m,t-13m,i} + c_4 Return_{t-60m,t-37m,i} + e_{t+1m,i},\end{aligned}$$

[4]We also run Fama–MacBeth cross-sectional regressions of models (F), (G), (H), and (I) using simulated data. The regression coefficients, t statistics, and R^2 are similar to the panel regression results.

$$\text{Model }(I): \quad Return_{t,t+1m,i} = d_0 + d_1 Growth\_S_{t,i} + d_2 Growth\_L_{t,i} + e_{t+1m,i},$$

where $Return_{t,t+1m,i}$ is the return from next month, $Return_{t-6m,t,i}$ is the six month cumulative return from time $t-6m$ to time t, $Return_{t-12m,t-7m,i}$ is the six month cumulative return from time $t-12m$ to time $t-7m$, $Return_{t-36m,t-13m,i}$ is two year cumulative return from time $t-36m$ to time $t-13m$, $Return_{t-60m,t-37m,i}$ is two year cumulative return from time $t-60m$ to time $t-37m$ for simulation i. In addition, the scaled average short-term growth rate is $Growth\_S_{t,i} := C_G \bar{G}_{S,i}(t)/((r+\alpha_D)(r+\alpha_S))$ and the scaled long-term growth rate is $Growth\_L_{t,i} := G_{L,i}(t)/((r+\alpha_D)(r+\alpha_L))$.

Table 1 illustrates that model (G) with two predictors obtains an R^2 similar to model (H) with five predictors. This implies that stock return from month $t-12m$ to month $t-7m$, stock return from month $t-36m$ to month $t-13m$, and stock return from month $t-60m$ to month $t-37m$ add little predictive power in predicting the next month return. Table 1 shows that the regression coefficients of a proxy for value effect $Value_{t,i}$ are significantly negative while the regression coefficients of a proxy for momentum effect $Return_{t-6m,t,i}$ are significantly positive. Consistent with our analytical result, Table 1 also shows that the regression coefficient of $Growth\_S_{t,i}$ is significantly positive and the regression coefficient of $Growth\_L_{t,i}$ is significantly negative.[5]

[5]Note that the t statistics are large in Table 1 because we run regression from 2000 independent simulations of monthly excess return over 40 years based on the return dynamics of one stock given in equation (38). The t statistics would be much smaller in a model with multiple stocks whose returns are correlated. The purpose of the t statistics is to establish that the simulation error is small, not to predict t statistics that would be obtained from actual, correlated data.

	Regression			
Variable	Model (F)	Model (G)	Model (H)	Model (I)
$Return_{t-6m,t,i}$	0.0167 (39.76)	0.0250 (57.61)	0.0257 (59.61)	
$Value_{t,i}$		-0.0184 (-70.10)	-0.0208 (-69.92)	
$Return_{t-12m,t-7m,i}$			0.0110 (23.62)	
$Return_{t-36m,t-13m,i}$			0.0037 (16.48)	
$Return_{t-60m,t-37m,i}$			0.0016 (7.72)	
$Growth\_S_{t,i}$				1.1047 (152.87)
$Growth\_L_{t,i}$				-0.0166 (-81.51)
R^2	0.0019	0.0109	0.0120	0.0334

Table 1—Panel Regression Results Using Simulated Data.

This table reports regression coefficients from panel regressions of models (F), (G), (H), and (I) using data from 2000 simulations of monthly excess return over 40 years based on the instantaneous excess return given in equation (38). Standard errors are clustered by simulation path and month, t statistics are in parentheses.