A Note on Preferences With Grades of Indecisiveness Without Reciprocity

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This note extends the analysis of Minardi and Savochkin (2013) by dropping the Reciprocity axiom. We provide a more general representation result and adapt the related analysis of comparative statics.

1 Introduction

This note extends the analysis of Minardi and Savochkin (2013) by providing a slightly more general representation that does not require imposing the Reciprocity axiom.[1] As we elaborate in the main paper, Reciprocity has normative appeal; however, it may seem somewhat restrictive from the descriptive viewpoint. In this note, we show that Reciprocity is far from being a crucial assumption in our earlier analysis, and that essentially the same representation can be obtained without it: A graded preference relation μ that satisfies the remaining axioms can be represented as

$$\mu(f,g) = \pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\right\}\right)$$

for a suitable von Neumann-Morgenstern utility function u, set of priors $\mathcal{M}$ and a capacity measure π that no longer need to satisfy the reciprocity

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[1] As a remainder, Reciprocity states that for any $f, g \in \mathcal{F}$ such that $\mu(f,g) < 1$, we have $\mu(g,f) = 1 - \mu(f,g)$.

condition — Condition (ii) of Theorem 1 in our earlier paper.

This generalization comes at the expense of making the statements of our axioms heavier. Indeed, without Reciprocity, the mere fact that $\mu(f,g) = 1$ for some acts f and g is not sufficient to conclude that the decision maker is absolutely sure that f is at least as good as g — that interpretation would be far-fetched if she simulteneously reports that $\mu(g,f) = 0.4$, for example. As in the previous paper, the combination of $\mu(f,g) = 1$ and $\mu(g,f) = 0$ represents a strict preference for f over g (for sure), while the combination of $\mu(f,g) = 1 = \mu(g,f)$ represents indifference. However, in the more general setting, we interpret the combination of $\mu(f,g) = 1$ and $\mu(g,f) > 0$ as an evidence that the decision maker is still indecisive.

The rest of the note has the following structure. We implicitly maintain the same setup as in the main paper, and use the same notation. In Section 2, we provide more general statements of our axioms, present a theorem showing the equivalence of the axioms and the representation, and repeat the comparative statics exercise. All proofs are contained in Section 3.

2 Main representation result

In what follows, we list the axioms that we impose on μ. Reflexivity and Independence have exactly the same formulation as in the main paper, while the remaining axioms need to be restated to take the lack of Reciprocity into account.

Axiom A1 (Reflexivity). For all $f \in \mathcal{F}$, $\mu(f,f) = 1$.

Axiom A2 (Weak Transitivity). For all $f,g,h \in \mathcal{F}$, if $\mu(f,g) = 1$ and $\mu(g,f) \in \{0,1\}$, then $\mu(f,h) \geq \mu(g,h)$ and $\mu(h,f) \leq \mu(h,g)$.

Axiom A3 (Monotonicity). For all $f,g \in \mathcal{F}$, if $\mu(f(\omega),g(\omega)) = 1$ and

$\mu(g(\omega), f(\omega)) \in \{0,1\}$ for all $\omega \in \Omega$, then $\mu(f,g) = 1$ and $\mu(g,f) \in \{0,1\}$.

Axiom A4 (C-Completeness). For all $x, y \in X$, $\mu(x,y) = 1$ and $\mu(y,x) = 0$, or $\mu(x,y) = 0$ and $\mu(y,x) = 1$, or $\mu(x,y) = 1$ and $\mu(y,x) = 1$.

Axiom A5 (Independence). For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1]$,

$$\mu(f,g) = \mu(\alpha f + (1-\alpha)h, \alpha g + (1-\alpha)h).$$

Axiom A6 (Continuity). For all $f, g, h \in \mathcal{F}$, the mappings $\alpha \mapsto \mu(\alpha f + (1-\alpha)g, h)$ and $\alpha \mapsto \mu(h, \alpha f + (1-\alpha)g)$ are continuous for all $\alpha \in [0,1]$, except the points at which $\mu(\alpha f + (1-\alpha)g, h) = 1 = \mu(h, \alpha f + (1-\alpha)g)$.

The above version of continuity directly postulates the analogue of the result of Lemma 8 from the main paper, which we derived there from the Continuity and Reciprocity axioms.

Axiom A7 (Nondegeneracy). $\mu(f,g) = 0$ and $\mu(g,f) = 1$ for some $f, g \in \mathcal{F}$.

Next, we re-introduce a property of capacity measures that is needed for our results.

Definition 1. We say that a capacity measure π on $\mathscr{B}(\mathcal{M})$, where $\mathcal{M}$ is a nonempty, closed, and convex subset of $\Delta(\Omega)$, has *linear full support* if, for any $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$, $\int_\Omega \varphi \, dp < 0$ for at least one $p \in \mathcal{M}$ implies that $\pi(\{p \in \mathcal{M} : \int_\Omega \varphi \, dp \geq 0\}) < 1$ or $0 < \pi(\{p \in \mathcal{M} : \int_\Omega \varphi \, dp \leq 0\}) < 1$.

The definition of linear continuity for a capacity measure maintains the same formulation as in the main paper, so we omit its restatement here.

Now, we state the main representation result for the case when μ does not necessarily satisfy Reciprocity.

Theorem 1. *Let μ be a graded preference relation on $\mathcal{F}$. Then, μ satisfies Reflexivity, Weak Transitivity, Monotonicity, C-Completeness, Independence, Continuity, and Nondegeneracy if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$, a nonempty, convex, and closed set $\mathcal{M}$ of probabilities in $\Delta(\Omega)$, and a linearly continuous capacity measure $\pi : \mathscr{B}(\mathcal{M}) \to [0,1]$ with linear full support such that, for all $f, g \in \mathcal{F}$,*

$$\mu(f,g) = \pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\right\}\right). \tag{1}$$

We conclude this subsection with a remark about Bewley's (1986) model of Knightian Uncertainty: Our results, showing that our model is its *refinement* (Propositions 4 and 5 of the main paper), still hold without Reciprocity if the notion of refinement is generalized as follows.

Definition 2. A graded preference μ is a *refinement* of a crisp binary relation $\succsim$ and $\succsim$ is a *coarsening* of μ if, for all $f, g \in \mathcal{F}$, the following conditions hold:

(i) $f \succ g \Leftrightarrow (\mu(f,g) = 1$ and $\mu(g,f) = 0)$; and
(ii) $f \sim g \Leftrightarrow (\mu(f,g) = 1$ and $\mu(g,f) = 1)$.

2.1 Comparative Attitudes

In this subsection, we extend the notion of one agent being "more decisive" than another which does not rely on Reciprocity, and provide a corresponding comparative statics result.

Definition 3. Given two graded preference relations μ_1 and μ_2, we say that μ_1 is *more decisive than* μ_2 if, for all $f, g \in \mathcal{F}$,

$$\mu_2(f,g) \geq \mu_2(g,f) \quad \Rightarrow \quad \mu_1(f,g) \geq \mu_2(f,g) \geq \mu_2(g,f) \geq \mu_1(g,f).$$

In words, whenever the second agent is inclined to prefer f over g, then the first (more decisive) agent has a (weakly) greater confidence that f

is indeed better than g in comparison with the second one, and, at the same time, has (weakly) less confidence that g is better than f, which, put differently, suggests that he is more confident that g is worse than f.

Our next result extends Theorem 7 in the main paper and provides a characterization of comparative indecisiveness in terms of representation (1).

Theorem 2. *Given two affine graded preferences μ_1 and μ_2 with representations $(u_1, \mathcal{M}_1, \pi_1)$ and $(u_2, \mathcal{M}_2, \pi_2)$, μ_1 is more decisive than μ_2 if and only if the following conditions hold:*

(i) u_1 *is a positive affine transformation of* u_2,

(ii) $\mathcal{M}_1 \subseteq \mathcal{M}_2$,

(iii) for all $B \in \mathfrak{H}_2$, $\pi_2(B) \geq \pi_2(\mathcal{M}_2 \backslash B)$ *implies* $\pi_1(\mathcal{M}_1 \bigcap B) \geq \pi_2(B)$, *and* $\pi_2(B) \leq \pi_2(\mathcal{M}_2 \backslash B)$ *implies* $\pi_1(\mathcal{M}_1 \bigcap B) \leq \pi_2(B)$.

3 Proofs

Lemma 3. *Suppose that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reflexivity, Weak Transitivity, Monotonicity, Independence, C-Completeness, Continuity, and Nondegeneracy axioms. Then, there exists a nonconstant affine function $u : X \to \mathbb{R}$ and a nonempty, convex, and closed set $\mathcal{M} \subseteq \Delta(\Omega)$ such that*

(i) for all $f, g \in \mathcal{F}$, $(\mu(f,g) = 1) \wedge (\mu(g,f) \in \{0,1\})$ *holds if and only if* $L_{f,g}p \geq 0$ *for all* $p \in \mathcal{M}$;

(ii) the closed and convex set $\mathcal{M}$ satisfying Part (i) is unique, and u is unique up to a positive affine transformation;

(iii) if $(u \circ f) - (u \circ g) = \lambda((u \circ f') - (u \circ g'))$ *for some* $\lambda > 0$ *and* $f, g, f', g' \in \mathcal{F}$, *then* $\mu(f,g) = \mu(f', g')$;

Proof. Step 1. An auxiliary crisp binary relation. Let $\succsim$ be a crisp binary relation on $\mathcal{F}$ defined as $f \succsim g \Leftrightarrow ((\mu(f,g) = 1) \wedge (\mu(g,f) \in \{0,1\}))$. We observe that $\succsim$ is reflexive, transitive, and has the monotonicity, independence, and nontriviality properties due to the Reflexivity, Weak Transitivity, Monotonicity, Independence, and Nontriviality axioms. Next, we prove that, for any $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed. To prove that the first of these sets is closed, suppose that the sequence $(\alpha_n)_{n=1}^{\infty}$ in $[0,1]$ converges to α and $\alpha_n f + (1-\alpha_n)g \succsim h$ for all $n \in \mathbb{N}$. If $\mu(\alpha f + (1-\alpha)g, h) = 1 = \mu(h, \alpha f + (1 - alpha)g)$, then $\alpha f + (1-\alpha)g \succsim h$ by definition. Otherwise, by the Continuity axiom, $\mu(\alpha_n f + (1-\alpha_n)g, h)$ converges to $\mu(\alpha f + (1-\alpha)g, h)$ as $n \to \infty$, and, therefore, $\mu(\alpha f + (1-\alpha)g, h) = 1$; similarly, $\mu(h, \alpha_n f + (1-\alpha_n)g)$ converges to $\mu(h, \alpha f + (1-\alpha)g)$ as $n \to \infty$, and, therefore, $\mu(h, \alpha f + (1-\alpha)g) \in \{0,1\}$. The closedness of the second set can be proven analogously.

Step 2. The von Neumann-Morgenstern utility function u. By C-Completeness, the restriction of $\succsim$ to X is a complete preorder. Therefore, by the Mixture Space Theorem (Herstein and Milnor, 1953), there exists an affine function $u : X \to \mathbb{R}$ such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Moreover, Nondegeneracy implies that u is nonconstant.

Step 3. The set $\mathcal{M}$ of priors. Define a binary relation $\tilde{\trianglerighteq}$ on $B_0(\Omega, \Sigma, u(X))$ as $(u \circ f) \tilde{\trianglerighteq} (u \circ g) \Leftrightarrow f \succsim g$ for all $f, g \in \mathcal{F}$. This binary relation is well defined, i.e., if $f', g' \in \mathcal{F}$ are such that $u \circ f = u \circ f'$ and $u \circ g = u \circ g'$, then $f \succsim g \Leftrightarrow f' \succsim g'$ by the Monotonicity axiom: Indeed, if $u \circ f = u \circ f'$ and $u \circ g = u \circ g'$, then $\mu(f(\omega), f'(\omega)) = 1 = \mu(f'(\omega), f(\omega))$ and $\mu(g(\omega), g'(\omega)) = 1 = \mu(g'(\omega), g(\omega))$ for all $\omega \in \Omega$, and, therefore, $f \sim f'$ and $g \sim g'$. Given the properties of $\succsim$, it is easy to check that $\tilde{\trianglerighteq}$ is a preorder that satisfies the nondegeneracy, archimedean continuity, monotonicity, and independence conditions of Gilboa, Maccheroni, Marinacci, and Schmeidler (2010, Appendix B). By their Corollary 1, there exists a nonempty, closed, and convex set $\mathcal{M} \subseteq \Delta(\Omega)$ such that

$$f \succsim g \quad \Leftrightarrow \quad \langle u \circ f, p \rangle \geq \langle u \circ g, p \rangle \text{ for all } p \in \mathcal{M};$$

moreover,

$$\mathcal{M} = \{p \in \Delta(\Omega) : L_{f,g}p \geq 0 \text{ for all } f,g \in \mathcal{F} \text{ such that } f \gtrsim g\}. \tag{2}$$

Claim (i) is now proven.

Claim (ii). Suppose that u' is a nonconstant affine function $X \to \mathbb{R}$ and $\mathcal{M}'$ is a closed and convex subset of $\Delta(\Omega)$ such that Claim (i) holds. By Nondegeneracy, $\mathcal{M}' \neq \varnothing$. Then, for any $x, y \in X$, $\mu(x,y) = 1 \Leftrightarrow u'(x) - u'(y) \geq 0$. Therefore, as follows from Herstein and Milnor (1953, Theorem 7), u' is a positive affine transformation of u. Finally, $\mathcal{M} = \mathcal{M}'$ by the uniqueness part of the same Corollary 1 of Gilboa et al. (2010).

Claim (iii). Suppose, first, that $f, f' \in \mathcal{F}$ are such that $u \circ f = u \circ f'$, and fix an arbitrary $g \in \mathcal{F}$. In this case, by the construction of u, we have $\mu(f(\omega), f'(\omega)) = 1 = \mu(f'(\omega), f(\omega))$ for all $\omega \in \Omega$. By Monotonicity, $\mu(f, f') = 1 = \mu(f', f)$. Then, $\mu(f, g) = \mu(f', g)$ and $\mu(g, f) = \mu(g, f')$ by Weak Transitivity.

Now, consider the general case: Suppose that $f, g, f', g' \in \mathcal{F}$ and $\lambda > 0$ are such that $(u \circ f) - (u \circ g) = \lambda((u \circ f') - (u \circ g'))$. Let $k \in int\, u(X)$ be chosen arbitrarily, and let $x \in X$ be such that $u(x) = k$ and $\varphi := (u \circ f) - (u \circ g)$. Then, one can find a sufficiently small $\varepsilon > 0$ such that $\lambda\varepsilon < 1$, and such that $\psi := \frac{1}{1-\varepsilon}u(x) - \frac{\varepsilon}{1-\varepsilon}(u \circ g)$ and $\psi' := \frac{1}{1-\lambda\varepsilon}u(x) - \frac{\lambda\varepsilon}{1-\lambda\varepsilon}(u \circ g')$ satisfy $\psi, \psi' \in u(X)^{\Omega}$. Let $h, h' \in \mathcal{F}$ be such that $\psi = u \circ h$ and $\psi' = u \circ h'$. We observe that

$$\begin{aligned} u \circ \big((1-\varepsilon)h + \varepsilon f\big) &= k + \varepsilon\varphi, & u \circ \big((1-\varepsilon)h + \varepsilon g\big) &= k, \\ u \circ \big((1-\lambda\varepsilon)h' + \lambda\varepsilon f'\big) &= k + \varepsilon\varphi, & u \circ \big((1-\lambda\varepsilon)h' + \lambda\varepsilon g'\big) &= k. \end{aligned}$$

Therefore, as follows from the claim proven in the preceding paragraph, $\mu((1-\varepsilon)h + \varepsilon f, (1-\varepsilon)h + \varepsilon g) = \mu((1-\lambda\varepsilon)h' + \lambda\varepsilon f', (1-\lambda\varepsilon)h' + \lambda\varepsilon g')$; at the same time, $\mu((1-\varepsilon)h + \varepsilon f, (1-\varepsilon)h + \varepsilon g) = \mu(f, g)$ and $\mu((1-\lambda\varepsilon)h' + \lambda\varepsilon f', (1-\lambda\varepsilon)h' + \lambda\varepsilon g') = \mu(f', g')$ by Independence, and we can conclude that $\mu(f, g) = \mu(f', g')$. □

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Lemma 4. *Suppose that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reflexivity, Weak Transitivity, Monotonicity, Independence, C-Completeness, Continuity, and Nondegeneracy axioms. Then, there exists a nonconstant affine function $u : X \to \mathbb{R}$ and a nonempty, convex, and closed set $\mathcal{M} \subseteq \Delta(\Omega)$ such that for any $f, g, f', g' \in \mathcal{F}$, $\{p \in \mathcal{M} : L_{f,g} p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{f',g'} p \geq 0\}$ implies that $\mu(f,g) \leq \mu(f',g')$.*

Proof. Step 1. Suppose, first, that $f, g, f', g' \in \mathcal{F}$ are such that $\{p \in \mathcal{M} : L_{f,g} p \geq 0\} \cap \{p \in \mathcal{M} : L_{f',g'} p \leq 0\} = \varnothing$. Fix an arbitrary $k \in int\, u(X)$, and find a sufficiently small $\varepsilon > 0$ such that, for

$$M := \max_{\omega \in \Omega} \max\{|u(f(\omega))|, |u(g(\omega))|, |u(f'(\omega))|, |u(g'(\omega))|\},$$

we have $[k - 2\varepsilon M, k + 2\varepsilon M] \subseteq u(X)$. Let $\varphi, \psi \in B(\Omega, \Sigma, \mathbb{R})$ be defined as $\varphi := (u \circ f) - (u \circ g)$ and $\psi := (u \circ f') - (u \circ g')$.

Step 2. We claim that there exists $\alpha \in (0,1)$ such that $\langle (1-\alpha)\varphi - \alpha\psi, p\rangle \leq 0$ for all $p \in \mathcal{M}$. Indeed, suppose, by contradiction, that, for each $\alpha \in (0,1)$, there exists $p \in \mathcal{M}$ such that $\langle (1-\alpha)\varphi - \alpha\psi, p\rangle > 0$. Let $C_1 := \{\xi \in B_0(\Omega, \Sigma, \mathbb{R}) : \langle \xi, p\rangle \leq 0$ for all $p \in \mathcal{M}\}$ and C_2 be the convex hull of 0, φ, and $-\psi$. Note that C_1 and C_2 are closed and convex sets; moreover, C_1 has a nonempty interior: The ball centered at the constant -1 of radius $\frac{1}{2}$ is contained in C_1 entirely. As follows from our assumption, $C_1 \cap \{(1-t)\varphi - t\psi \mid t \in (0,1)\} = \varnothing$, and, therefore, $int\, C_1 \cap C_2 = \varnothing$. Then, by the Interior Separating Hyperplane theorem (Aliprantis and Border, 2006, Theorem 5.67), there exists a nonzero continuous linear functional L^0 on $B_0(\Omega, \Sigma, \mathbb{R})$ such that $L^0 \xi \leq 0$ for all $\xi \in C_1$ and $L^0 \xi \geq 0$ for all $\xi \in C_2$. This L^0 can be represented as $L^0 \xi = \langle \xi, p\rangle$ for all $\xi \in B_0(\Omega, \Sigma, \mathbb{R})$, where p is some nonzero, bounded, and finitely additive function $\Sigma \to \mathbb{R}$ (Aliprantis and Border, 2006, Lemma 14.31). Notice that $p(E) \geq 0$ for all $E \in \Sigma$ because $-\mathbb{1}_E \in C_1$; therefore, we can assume without loss of generality that $p \in \Delta(\Omega)$. Then, in fact, $p \in \mathcal{M}$, as follows from Step 3 of the proof of Lemma 9. This means that we have found $p \in \mathcal{M}$ such that $\langle \varphi, p \rangle \geq 0$ and $\langle \psi, p \rangle \leq 0$, a contradiction to the assumption made in Step 1.

Step 3. Let $f^*, g^*, h^* \in \mathcal{F}$ be such that $u \circ f^* \equiv k$, $u \circ g^* = k + \varepsilon(1-\alpha)\varphi - \varepsilon\alpha\psi$, $u \circ h^* = k - \varepsilon\alpha\psi$, and note that such acts exist by the choice of ε. By the result of the Step 2, we have $L_{f^*,g^*}p \geq 0$ for all $p \in \mathcal{M}$, and, therefore, $(\mu(f^*, g^*) = 1 \wedge (\mu(g^*, f^*) \in \{0, 1\}))$. By Weak Transitivity, this implies that $\mu(f^*, h^*) \geq \mu(g^*, h^*)$. Now, observe that $(u \circ f^*) - (u \circ h^*) = \varepsilon\alpha\psi = \varepsilon\alpha[(u \circ f') - (u \circ g')]$ and $(u \circ g^*) - (u \circ h^*) = \varepsilon(1-\alpha)\varphi = \varepsilon(1-\alpha)[(u \circ f) - (u \circ g)]$. By Part (iii) of Lemma 9, we have $\mu(f, g) = \mu(g^*, h^*)$ and $\mu(f^*, h^*) = \mu(f', g')$, which proves (under the assumption of Step 1) that $\mu(f, g) \leq \mu(f', g')$.

Step 4. Now, let f, g, f', g' be arbitrary acts such that $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$. Fix an arbitrary $k \in int\, u(X)$, let $x_0, x_1 \in X$ be such that $u(x_0) = k$ and $u(x_1) > u(x_0)$, and $f'', g'' \in \mathcal{F}$ be defined as $f'' := \frac{1}{2}x_0 + \frac{1}{2}f'$ and $g'' := \frac{1}{2}x_0 + \frac{1}{2}g'$, and observe that $(u \circ f'') - (u \circ g'') = \frac{1}{2}[(u \circ f') - (u \circ g')$, and, therefore, $\{p \in \mathcal{M} : L_{f'',g''}p \geq 0\} = \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$. Finally, let $f''_t := \frac{1}{2}(1-t)x_0 + \frac{1}{2}tx_1 + \frac{1}{2}f'$ for all $t \in [0, 1]$, and note that $f''_0 = f''$.

We claim that $\{p \in \mathcal{M} : L_{f'',g''}p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{f''_t,g''}p \geq 0\}$ and $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \cap \{p \in \mathcal{M} : L_{f''_t,g''}p \leq 0\} = \varnothing$ for all $t \in (0, 1]$. Indeed, for all $p \in \Delta(\Omega)$, we have $L_{f''_t,g''}p = L_{f'',g''}p + \frac{1}{2}t[u(x_1) - u(x_0)]$, and, therefore, $L_{f'',g''}p \geq 0$ implies $L_{f''_t,g''}p > 0$.

Given the above no-intersection property, we have $\mu(f, g) \leq \mu(f''_t, g'')$ for all $t \in (0, 1]$, as proven earlier. By Continuity, this implies $\mu(f, g) \leq \mu(f'', g'')$. Since $\mu(f'', g'') = \mu(f', g')$ by Part (iii) of Lemma 9, the proof of the lemma is now complete. $\square$

Proof of Theorem 1. *Only if* part. Assume that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reflexivity, Weak Transitivity, Monotonicity, C-Completeness, Independence, Continuity, and Nondegeneracy axioms.

Step 1. Let function $u : X \to \mathbb{R}$ and set $\mathcal{M} \subseteq \Delta(\Omega)$ be as described in Lemma 3. Let $K := u(X)$, $\mathfrak{H} := \{\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \mid f, g \in \mathcal{F}\}$, and let

$\pi : \mathfrak{H} \to [0,1]$ be defined as $\pi(\{p \in \mathcal{M} : L_{f,g}p \geq 0\}) := \mu(f,g)$. By Lemma 4, such π is well defined: If $f, g, f', g' \in \mathcal{F}$ are such that $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} = \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$, then it must be that $\mu(f,g) = \mu(f',g')$. By the same lemma, π is monotone, and it is normalized by construction.

Step 2. To prove the linear continuity of π, consider an arbitrary nonconstant $L \in \mathcal{L}$, and suppose that $Lp = \langle \varphi, p \rangle$, where $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$.

Let $k \in int\, K$ be chosen arbitrary, and let $\varepsilon > 0$ be a sufficiently small number such that $[k - 3\varepsilon\|\varphi\|, k + 3\varepsilon\|\varphi\|] \subset K$. Let $\xi := k + \varepsilon\varphi$, and note that $Lp = \frac{1}{\varepsilon}\langle \xi - k, p\rangle$ for all $p \in \mathcal{M}$. Our objective is to prove that the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(\alpha) := \pi(\{p \in \mathcal{M} : \langle \xi - k, p \rangle \geq \alpha\})$ is continuous.

First, let $\underline{\alpha} := \sup\{\alpha \in \mathbb{R} : \forall_{p \in \mathcal{M}} \langle \xi - k, p\rangle \geq \alpha\}$, $\overline{\alpha} := \inf\{\alpha \in \mathbb{R} : \forall_{p \in \mathcal{M}} \langle \xi - k, p\rangle < \alpha\}$, and observe that F is monotone, $F(\alpha) = 1$ for all $\alpha < \underline{\alpha}$, and $F(\alpha) = 0$ for all $\alpha > \overline{\alpha}$.

Second, we note that $\underline{\alpha} > -2\varepsilon\|\varphi\|$ and $\overline{\alpha} < 2\varepsilon\|\varphi\|$, and, therefore, for any $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, we have $\xi - \alpha \in K^\Omega$ by the choice of ε. Hence, there exist $x \in X$ and $f, g \in \mathcal{F}$ such that $k = u(x)$, $\xi - \underline{\alpha} = u \circ f$, and $\xi - \overline{\alpha} = u \circ g$. Let $f_\alpha := \frac{\overline{\alpha} - \alpha}{\overline{\alpha} - \underline{\alpha}} f + \frac{\alpha - \underline{\alpha}}{\overline{\alpha} - \underline{\alpha}} g$, and note that the mapping $\alpha \mapsto f_\alpha$ is continuous on $[\underline{\alpha}, \overline{\alpha}]$. It is easy to verify that $u \circ f_\alpha = \xi - \alpha$, and, thus, $F(\alpha) = \pi(\{p \in \mathcal{M} : \langle \xi, p \rangle - \alpha - k \geq 0\}) = \mu(f_\alpha, x)$ for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

Finally, we observe that there is no $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ such that $\mu(f_\alpha, x) = 1 = \mu(x, f_\alpha)$: Otherwise, as follows from Claim (i) of Lemma 3, we would have $\langle \varepsilon\varphi - \alpha, p\rangle = 0$ for all $p \in \mathcal{M}$, which contradicts to L being nonconstant. Therefore, $\mu(f_\alpha, x)$ is continuous in α by the Continuity axiom, and the continuity of F is proven.

Step 3. To prove that π has linear full support, suppose that $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$ is such that $\langle \varphi, q \rangle < 0$ for some $q \in \mathcal{M}$. Let $k \in int\, K$ be arbitrary, and $\varepsilon > 0$ be a sufficiently small number such that $k + \varepsilon\varphi \in K^\Omega$. Finally, let $x \in X$ and $f \in \mathcal{F}$ be such that $u(x) = k$ and $u \circ f = k + \varepsilon\varphi$. By Claim (i) of Lemma 3, $L_{f,x}q < 0$ implies that $\mu(f,x) < 1$ or $\mu(x,f) \notin \{0,1\}$.

Step 4. Now, we extend π from $\mathfrak{H}$ to $\mathscr{B}(\mathcal{M})$. For any $\mathcal{S} \in \mathscr{B}(\mathcal{M})$, let $\hat{\pi}(\mathcal{S}) := \sup\{\pi(\mathcal{S}') \mid \mathcal{S}' \in \mathfrak{H}, \mathcal{S}' \subseteq \mathcal{S}\}$. As follows from the monotonicity of π, $\hat{\pi}$ agrees with π on the intersection of their domains: $\hat{\pi}(S) = \pi(S)$ for all $\mathcal{S} \in \mathfrak{H}$. Therefore, Representation (1) holds for $\hat{\pi}$, as well. Monotonicity and normalization of $\hat{\pi}$ follow from the monotonicity of π. Finally, linear continuity and linear full support of $\hat{\pi}$ hold immediately because these properties operate with the values of $\hat{\pi}$ on sets that always belong to $\mathfrak{H}$.

If part. Assume that there exist a nonconstant affine function $u : X \to \mathbb{R}$, a nonempty, convex, and closed set $\mathcal{M} \subseteq \Delta(\Omega)$, and a capacity $\pi : \mathscr{B}(\mathcal{M}) \to [0,1]$ such that Representation (1) holds. As directly follows from (1), μ satisfies Reflexivity, Monotonicity, C-Completeness, and Independence. Nondegeneracy of μ follows from u being nonconstant.

To prove Weak Transitivity, suppose $f, g \in \mathcal{F}$ are such that $\mu(f, g) = 1$ and $\mu(g, f) \in \{0, 1\}$. Let $h \in \mathcal{F}$ be arbitrary. Since π has linear full support, it follows that $\langle u \circ f, p\rangle \geq \langle u \circ g, p\rangle$ for all $p \in \mathcal{M}$. Consequently, $\{p \in \mathcal{M} : L_{f,h}p \geq 0\} \supseteq \{p \in \mathcal{M} : L_{g,h}p \geq 0\}$, and $\{p \in \mathcal{M} : L_{h,g}p \geq 0\} \supseteq \{p \in \mathcal{M} : L_{h,f}p \geq 0\}$. By the monotonicity of π, we have $\mu(f, h) \geq \mu(g, h)$ and $\mu(h, g) \geq \mu(h, f)$.

To prove Continuity, fix $f, g, h \in \mathcal{F}$ and $\alpha \in [0, 1]$ such that $\mu(\alpha f + (1 - \alpha)g, h) < 1$ or $\mu(h, \alpha f + (1 - \alpha)g) < 1$, and consider a sequence $(\alpha_n)_{n=1}^{\infty}$ of numbers in $[0, 1]$ such that $\alpha_n \to \alpha$. Let $M := \sup_{\omega\in\Omega} |u(f(\omega)) - u(g(\omega))|$, and notice that $M < \infty$ because f and g take only finitely many values. Then,

$$\begin{aligned}\Big\|\big(u \circ (\alpha_n f + (1 - \alpha_n)g)\big) - \big(u \circ (\alpha f + (1 - \alpha)g)\big)\Big\| = \\ \|(\alpha_n - \alpha)(u \circ f) - (\alpha_n - \alpha)(u \circ g)\| \leq |\alpha_n - \alpha| \cdot M.\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$L_{\alpha f+(1-\alpha)g,h}p - \varepsilon \leq L_{\alpha_n f+(1-\alpha_n)g,h}p \leq L_{\alpha f+(1-\alpha)g,h}p + \varepsilon$$

for all $p \in \Delta(\Omega)$. Hence,

$$\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq \varepsilon\} \subseteq \{p \in \mathcal{M} : L_{\alpha_n f+(1-\alpha_n)g,h} p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq -\varepsilon\}$$

for all $n > N$, which implies that

$$\pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq \varepsilon\}) \leq \pi(\{p \in \mathcal{M} : L_{\alpha_n f+(1-\alpha_n)g,h} p \geq 0\}) \leq \pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq -\varepsilon\})$$

for all $n > N$ by the monotonicity of π. Thus, we obtain

$$\begin{aligned}
&\pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq \varepsilon\}) \leq \\
&\quad \liminf_{n\to\infty} \pi(\{p \in \mathcal{M} : L_{\alpha_n f+(1-\alpha_n)g,h} p \geq 0\}) \leq \\
&\quad \limsup_{n\to\infty} \pi(\{p \in \mathcal{M} : L_{\alpha_n f+(1-\alpha_n)g,h} p \geq 0\}) \leq \\
&\qquad \pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq -\varepsilon\})
\end{aligned} \tag{3}$$

for all $\varepsilon > 0$. Now, we observe that $L_{\alpha f+(1-\alpha)g,h}$ cannot be a zero constant on $\mathcal{M}$ due to the assumption that $\mu(\alpha f + (1-\alpha)g, h) < 1$ or $\mu(h, \alpha f + (1-\alpha)g) < 1$. If $L_{\alpha f+(1-\alpha)g,h}$ is a positive constant on $\mathcal{M}$, then $\pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq \varepsilon\}) = 1 = \pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq -\varepsilon\})$ for all sufficiently small positive ε; similarly, if $L_{\alpha f+(1-\alpha)g,h}$ is a negative constant on $\mathcal{M}$, then $\pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq \varepsilon\}) = 0 = \pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq -\varepsilon\})$ for all sufficiently small positive ε. Finally, if $L_{\alpha f+(1-\alpha)g,h}$ is nonconstant on $\mathcal{M}$, then, by the linear continuity of π, the left-most and the right-most parts of (3) converge to $\pi(\{p \in \mathcal{M} : L_{\alpha f+(1-\alpha)g,h} p \geq 0\})$. This all proves that $\lim_{n\to\infty} \mu(\alpha_n f + (1-\alpha_n)g, h) = \mu(\alpha f + (1-\alpha)g, h)$. The symmetric equality can be proven similarly. □

Lemma 5. *Suppose μ is an affine graded preference relation with the representation $(u, \mathcal{M}, \pi)$, $f, g \in \mathcal{F}$ are such that there exists $p \in \mathcal{M}$ such that $\langle u \circ f, p\rangle \neq \langle u \circ g, p\rangle$ for some $p \in \mathcal{M}$, and let $B := \left\{p \in \mathcal{M} : \int (u \circ f)\, dp \geq \int (u \circ g)\, dp\right\}$. Then, $\pi(\mathcal{M} \backslash B) = \mu(g, f)$.*

Proof. First, we consider the case that $L_{f,g}$ is constant on $\mathcal{M}$ — i.e., there exists $c \in \mathbb{R}\backslash\{0\}$ such that $L_{f,g}p = c$ for all $p \in \mathcal{M}$. If $c > 0$, then $\mathcal{M}\backslash B = \{p \in \mathcal{M} : L_{f,g}p < 0\} = \varnothing = \{p \in \mathcal{M} : L_{f,g}p \leq 0\}$, and $\pi(\mathcal{M}\backslash B) = \mu(g, f)$. Similarly, if $c < 0$, then $\mathcal{M}\backslash B = \{p \in \mathcal{M} : L_{f,g}p < 0\} = \mathcal{M} = \{p \in \mathcal{M} : L_{f,g}p \leq 0\}$, and $\pi(\mathcal{M}\backslash B) = \mu(g, f)$. Finally, if $L_{f,g}$ is nonconstant on $\mathcal{M}$, then, by the monotonicity of π, we observe $\limsup_{\varepsilon\to 0+} \pi(\{p \in \mathcal{M} : L_{g,f}p \geq \varepsilon\}) \leq \pi(\{p \in \mathcal{M} : L_{g,f}p > 0\}) \leq \pi(\{p \in \mathcal{M} : L_{g,f}p \geq 0\})$, which implies that $\pi(\{p \in \mathcal{M} : L_{g,f}p > 0\}) = \pi(\{p \in \mathcal{M} : L_{g,f}p \geq 0\})$ by the linear continuity of π. ☐

Proof of Theorem 2. *Only if* part. Suppose μ_i, where $i = 1, 2$ are affine graded preferences with the representations $(u_i, \mathcal{M}_i, \pi_i)$, and μ_1 is more decisive than μ_2. Let $\succsim_i$, where $i = 1, 2$, be binary relations on $\mathcal{F}$ defined as $f \succsim_i g \Leftrightarrow (\mu(f, g) = 1) \wedge (\mu(g, f) \in \{0, 1\})$. Since μ_1 is more decisive than μ_2, we have $\succsim_2 \subseteq \succsim_1$. By Claim (i) of Lemma 3, binary relations $\succsim_i$, where $i = 1, 2$, admit representations $f \succsim_i g \Leftrightarrow (\forall_{p\in\mathcal{M}_i} \langle u_i \circ f, p\rangle \geq \langle u_i \circ g, p\rangle)$. Then, Claims (i) and (ii) follow from the proof of Proposition 6 of Ghirardato, Maccheroni, and Marinacci (2004).

We now turn to proving Claim (iii). Fix an arbitrary $B \in \mathfrak{H}_2$; by the definition of $\mathfrak{H}_2$, there exist $f, g \in \mathcal{F}$ such that $B = \left\{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\right\}$.

Suppose that $\pi_2(B) \geq \pi_2(\mathcal{M}_2\backslash B)$. We observe, first, that $\mu_2(f, g) \geq \mu_2(g, f)$: Indeed, if $\langle u_2 \circ f, p\rangle = \langle u_2 \circ g, p\rangle$ for all $p \in \mathcal{M}_2$, then $\mu_2(g, f) = 1 = \mu_2(f, g)$; otherwise, by Lemma 5, $\mu_2(g, f) = \pi(\mathcal{M}_2\backslash B) \leq \pi(B) = \mu_2(f, g)$. Then, $\mu_2(f, g) \leq \mu_1(f, g)$ because μ_1 is more decisive, and we obtain $\pi_2(B) = \mu_2(f, g) \leq \mu_1(f, g) = \pi_1(B_1)$, where $B_1 := \left\{p \in \mathcal{M}_1 : \int (u_1 \circ f)\, dp \geq \int (u_1 \circ g)\, dp\right\}$. Finally, we observe that B_1 is equal to $\left\{p \in \mathcal{M}_1 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\right\}$ by Claim (i), and, in turn, to $\mathcal{M}_1 \cap B$ by Claim (ii). Therefore, we have $\pi_2(B) \leq \pi_1(\mathcal{M}_1 \bigcap B)$.

If $B \in \mathfrak{H}_2$ is such that $\pi_2(B) \leq \pi_2(\mathcal{M}_2\backslash B)$, the facts that $\mu_2(f, g) \leq \mu_2(g, f)$ and, in turn, $\pi_2(B) \geq \pi_1(\mathcal{M}_1 \bigcap B)$ can be proven by an argument symmetric to the one for the previous case.

If part. Suppose that Conditions (i), (ii), and (iii) of the proposition hold, and fix arbitrary $f, g \in \mathcal{F}$.

If $\mu_2(f,g) \geq \mu_2(g,f)$, then

$$\pi_2\Big(\Big\{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\Big\}\Big) \geq \\ \pi_2\Big(\Big\{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp < \int (u_2 \circ g)\, dp\Big\}\Big).$$

This follows from Lemma 5 if $\langle u_2 \circ f, p\rangle \neq \langle u_2 \circ g, p\rangle$ for some $p \in \mathcal{M}_2$, and is immediate otherwise, given that $\pi_2(\mathcal{M}_2) = 1$ and $\pi_2(\varnothing) = 0$. Then,

$$\begin{aligned} \pi_2(B) &\leq \pi_1(\mathcal{M}_1 \cap B) && \text{by (iii)} \\ &= \pi_1\big(\big\{p \in \mathcal{M}_1 : \textstyle\int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\big\}\big) && \text{by (ii)} \\ &= \pi_1\big(\big\{p \in \mathcal{M}_1 : \textstyle\int (u_1 \circ f)\, dp \geq \int (u_1 \circ g)\, dp\big\}\big) && \text{by (i).} \end{aligned}$$

Therefore, we have $\mu_2(f,g) \leq \mu_1(f,g)$.

If $\mu_2(f,g) \leq \mu_2(g,f)$, then

$$\pi_2\Big(\Big\{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\Big\}\Big) \leq \\ \pi_2\Big(\Big\{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp < \int (u_2 \circ g)\, dp\Big\}\Big)$$

follows, again, from Lemma 5 (and the case that $\langle u_2 \circ f, p\rangle = \langle u_2 \circ g, p\rangle$ for all $p \in \mathcal{M}_2$ is now impossible). Given that, the proof that $\mu_2(f,g) \geq \mu_1(f,g)$ proceeds symmetrically to the previous case. □

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