Mistake Aversion and a Theory of Robust Decision Making*

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This paper studies the behavioral trait of aversion to making mistakes. I use the framework of subjective uncertainty to develop a choice procedure that captures mistake aversion along the lines of robust decision making: For each probability distribution (prior) that may plausibly describe the underlying uncertainty, the decision maker computes his expected utility and discards the feasible choice options that cannot guarantee a particular level of utility relative to an exogenously given default option. The paper then follows the revealed preference approach to study the behavioral foundations of this procedure and the comparative notion of mistake aversion. In an application to finance, I show that mistake aversion leads to an increase of the risk premia. The effects of the volatility of payoffs on asset prices/returns are also found to be potentially different than in the predictions of the standard expected utility theory and the maxmin model of ambiguity aversion.

1 Introduction

The standard subjective expected utility paradigm postulates that an economic agent, when facing uncertainty with unspecified probabilities, brings forth his subjective probability measure and, then, ranks alternatives by the mathematical expectation of his utility function computed with respect to

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that measure. However, it is now well known that this paradigm falls short of explaining much of the empirical observations found in experimental studies. Moreover, even when agents can plausibly formulate a *single* prior to evaluate likelihoods of uncertain events, their aptitude for risk taking cannot always be explained fully by the concavity of their utility function. The limited descriptive power of the subjective expected utility theory manifests well beyond the lab, and issues such as the equity premium puzzle are frequently addressed by using more-permissive models of preferences that allow for ambiguity aversion, habits, preference for flexibility, and other phenomena.

This work is motivated by a behavioral trait that has not been explicitly studied in literature so far: the decision maker's concern about making mistakes. The prevalence of this concern is self-evident; this paper, however, restricts attention to the domain of uncertainty and focuses on a particular type of mistake that consists of using a wrong prior in the ex ante decision, which results in a substantial loss.

Thus, the ultimate goal of this work is to understand how to model cautious choice under uncertainty that is driven by mistake aversion. To address this issue, the paper studies both the formal behavioral patterns that distinguish mistake aversion from other types of cautious behavior, including standard risk aversion, and the procedural deviations from expected utility maximization.

The choice procedure proposed in this paper follows the general approach of robust decision making and ensures that inaccuracies in estimating the probabilities of events do not lead to mistakes: First, for each alternative probability distribution that may plausibly describe the uncertainty, the agent computes his expected utility and discards those feasible choice options that cannot guarantee a particular level of utility relative to his default option (or another salient safe choice). Second, among the remaining alternatives, he chooses the one that maximizes his expected utility computed with respect to his leading prior.

The main theoretical result of this work consists of establishing the axiomatic foundations of the proposed procedure. The foundations help to ensure that the procedure is consistent with the objective of modeling mistake aversion, to clarify its scope, to differentiate it from other models, and to potentially test it in the lab. In addition, the paper defines and characterizes the notion of "being more mistake-averse than," which is needed for comparative statics and may be useful for calibrating the model and interpreting both its predictions and its implied policy recommendations. Finally, the paper contains a small example that demonstrates the usage of the model in a general equilibrium setting and illustrates possible implications of mistake aversion for asset prices.

Now, a more detailed description of the proposed choice procedure is in order. Suppose that uncertainty is represented by a nonempty finite space Ω of the states of nature and that the alternatives are represented by vectors of state-contingent payoffs (traditionally called *acts*). The decision problem that the agent faces is to pick the best alternative from a finite set of such vectors.

As a part of the framework, I also assume that each set of alternatives is presented to the decision maker with one option marked as the *default*; moreover, this default represents a course of action that is always viewed as a safe choice. Therefore, each choice problem represents a situation in which, loosely speaking, "higher returns" are associated with higher risks, and the decision maker is asked to decide whether he prefers the safe option or is willing to take some risk in exchange for the prospect of a better outcome.

My assumptions about the default also imply that, in each choice set, the analyst can identify one option that is unquestionably safe. Such a safe option can be identified relatively naturally in many potential applications, such as the savings problem (in which the agent can consume his endowment); portfolio choice (investment at the risk-free rate rather than risky assets); bargaining and contracts (taking the outside option); and job search (maintaining the current employment status and employer), just to name a few.[1]

Thus, the problem that the decision maker faces is to identify his preferred alternatives from a nonempty finite set of acts S (the feasible set), given a safe option d in S. In my model, the decision maker is described by a utility function u taking values in $\mathbb{R}_+$; a prior p over Ω; a set $\mathcal{M}$ of probability distributions over Ω that represents the collection of alternative scenarios that he considers; and, finally, a "tolerance" function $k : \mathcal{M} \to [0,1]$. His choice, then, is obtained by solving the following optimization problem:

$$\begin{aligned}
&\underset{f \in S}{\text{Maximize}} \sum_{\omega \in \Omega} u(f(\omega))\, p(\omega) \\
&\text{s.t. } \sum_{\omega \in \Omega} u(f(\omega))\, q(\omega) \geq k(q) \sum_{\omega \in \Omega} u(d(\omega))\, q(\omega) \text{ for all } q \in \mathcal{M}.
\end{aligned}$$

This choice procedure can be viewed as a two-stage process. In the first stage, captured by the constraints, the decision maker eliminates from his feasible set all alternatives that may be potential mistakes. An alternative becomes a mistake if choosing it leads to the loss of the agent's expected utility relative to the expected utility of the safe option, and that loss exceeds a certain threshold. The expectations in this stage are computed using all plausible scenarios that are represented by the corresponding probability distributions, and the alternative is discarded if it can be regarded as a mistake under at least one scenario. In the second stage, the decision maker maximizes his expected utility among the alternatives that have not been

[1] Indeed, there is now a growing literature on the theory of choice in the presence of status quo, default options, or anchors — this literature will be discussed later in Section 2.

eliminated in the first stage.[2] Note that the default option always passes the first stage, so the maximization is done over a nonempty set, and the decision maker's choice is always well-defined.

The function $k(q)$ in the representation can be interpreted as the decision maker's *tolerance* to losses: If, for example, $k(q_0) = 0.95$ for some q_0 in $\mathcal{M}$, it means that the decision maker can tolerate, without experiencing overwhelming negative feelings, the situation in which he discovers that the right probability distribution to evaluate the likelihoods of the states is q_0 but not p, and he loses 5% of his (expected) wealth as a result. In addition, as elaborated later, the function k also embodies the decision maker's opinion about the *relevance* of various probability distributions. The standard expected utility preferences can be viewed as a special case of this model if we let $k(q) = 0$ for all q (or if $\mathcal{M} = \varnothing$).

The main result of this paper shows that the choice procedure described above arises from a relatively simple set of axioms. The first axiom is a version of the Weak Axiom of Revealed Preference that is adapted to a setting with an exogenous default. The second axiom is a restricted version of the classical Anscombe-Aumann Independence axiom that covers only the situations in which the alternatives in question are undoubtedly not potential mistakes. The third axiom is scale invariance, which is akin to positive homogeneity and ensures that the agent's preferences remain unaffected if the payoff profiles are modified in such a way that the utility levels in all states are multiplied by a constant. As an implication, if, in addition, the agent's utility function over risk has the constant relative risk aversion property, this preserves the overall independence of choice from the scale. Finally, I have two novel axioms that describe situations in which the agent may justify his choice of a non-default alternative in one situation by referring

[2]If necessary, the model can also be extended to include a non-expected utility functional to be maximized in the second stage.

to another choice situation, in which the agent chose a related non-default alternative and, hence, determined that it was not a potential mistake.[3] These axioms, together with standard technical conditions, are found necessary and sufficient to conclude that the agent chooses in agreement with the proposed procedure.

As an application, I illustrate the implications of mistake aversion for asset prices in a simple endowment economy. The economy consists of two agents, one of whom has standard expected utility preferences, and two assets — one riskless and one with log-normal gross payoffs. The presence of mistake aversion leads to a noticeable departure of equilibrium prices from the predictions of the standard expected utility theory and the maxmin model of ambiguity aversion. First, my economy has a higher risk premium in comparison with a similar economy populated only by expected utility maximizers. Second, mistake aversion changes the way the agents' willingness to take risks depends on the volatility of the risky asset's payoffs. The effects of mistake aversion can be summarized as follows: If the agents' risk aversion is estimated using the data collected in an environment in which the volatility of payoffs is low, then the equilibrium risk premium of medium-volatility assets will be higher than the predictions of the standard expected utility model. At the same time, in high-volatility environments, the agents are willing to take more risk than can be expected solely on the basis of the risk aversion that they manifest in medium-volatility environments. Such a double-sided effect does not arise in the maxmin model. In addition, in the

[3] One justification of this kind may be described as follows: Suppose that an act A was chosen over a default D, A can be represented as a mixture of a and some other act A', and b dominates a in every possible state of the world. Then, it is not a mistake to choose over D an act B obtained from A by replacing a with b in the mixture. Two axioms stated in Section 4.2 describe more general situations in which a similar intuition applies.

range of low to medium volatilities, my model predicts a higher sensitivity of the volatility of asset prices to the volatility of payoffs.

The rest of the paper is organized as follows. In Section 2, I discuss the related literature and compare my model with the existing models of regret, anchoring and status quo bias, and ambiguity. In Section 3, I illustrate the model by computing equilibrium equity premiums in an economy populated by expected utility and mistake-averse agents. The mathematically precise description of the model, its axiomatic foundations, and comparative statics results are provided in Section 4. All proofs are gathered in the appendices.

2 Related Literature

This research is related to three broad branches of literature — the theory of regret; the theory of anchoring, including status quo bias and endowment effects; and robust decision making, including the theory of ambiguity.

Theory of regret The literature on regret, starting with the seminal work of Savage (1951) and continued by Bell (1982) and Loomes and Sugden (1982), as well as recent contributions of Hayashi (2008) and Stoye (2011), studies agents who are subject to regret, "... a negative, cognitively based emotion that we experience when realizing or imagining that our present situation would have been better, had we decided differently."[4] The decision-theoretic analysis then focuses on the ex ante behavior of agents who are forward-looking, understand that many of their choices will lead to ex post regret, and want to minimize that feeling.

The structure of the models in this literature implies that the agent may experience regret in quite a few states and decision problems, and his negative emotion is based on the implicit assumption that he could have perfectly

[4] Zeelenberg (1999, p. 94).

predicted the realized state.[5] In contrast, in my work, the agent does not suffer psychological penalties from his inability to predict the state, but he is concerned about having a flawed *basis* for his choice — a wrong probability law for computing his expected utility.

Another notable difference between the classical theory of regret and my model involves internal consistency of choices. The majority of models in that literature exhibit the lack of transitivity, while the choice procedure proposed in this paper does not have this issue.[6]

Theories of anchoring effects The setup of my work — choice from a set given a default option — connects it to the choice-theoretic literature on modeling the framing effect, which follows the general framework proposed by Salant and Rubinstein (2008) and covers the status quo bias and the effects of endowment, default option, and other types of anchoring.

Masatlioglu and Ok's (2013) canonical model of choice with status quo represents the agent's choice as $c(S,d) = \operatorname{Arg\,max}_{x \in S \cap Q(d)} U(x)$, where U is the agent's utility function and Q is a nonempty-valued correspondence that determines the agent's mental constraints. The choice procedure that I propose follows this general form. At the same time, the effect of the default option is not a primary object of study in this work. Rather, its presence serves as a tool to understand better how the agent makes choices that are

[5] For example, a decision maker who follows the classical Savage's model would feel excruciating regret after hearing that 10, 13, 14, 22, 52 and 11 were the winning numbers of the the Powerball drawing on May, 18th, 2013, the jackpot of which totaled almost 600 million dollars. The reason is that he could have chosen exactly these numbers, but did not, either choosing different numbers or not playing on that day. Such ex post suboptimality of actions is exactly what is assumed to cause negative emotions; in turn, anticipation of such emotions is what drives the agent's decisions at the ex ante stage in those models.

[6] Indeed, as will become clear in Section 4, my model satisfies a version of the Weak Axiom of Revealed Preference.

robust to possible inaccuracies in his probabilistic model of the world.

Departing from the popular technique of studying an agent by comparing his choices from the same set with and without the presence of the status quo,[7] the results of this paper rely only on choices in the presence of a default option, thus making fewer assumptions about what is observable. Also, my choice procedure is not consistent with some of the key axioms proposed in this literature, such as the Status Quo Bias axiom of Masatlioglu and Ok (2005) (also used in Ortoleva, 2010), Strong Axiom of Status Quo Bias of Masatlioglu and Ok (2013), and Sagi's (2006) No-Regret condition. Many of these axioms lose their normative appeal when the "decision frame" d is interpreted as a blame-free reference point rather than an element of the choice set that gains in attractiveness because of its current status as an endowment or status quo.

Two papers in this branch of literature — Ortoleva (2010) and Riella and Teper (2012) — apply the ideas of status quo bias to the domain of uncertainty, which makes their representations easier to compare with mine. Ortoleva's (2010) model does not put structure on the agent's choices among options that are not ruled out by the presence of the status quo, whereas I assume that the decision maker maximizes his expected utility in that subdomain. However, the psychological effects of the status quo in his model have a much simpler form: Barring other differences,[8] they correspond to the special case of my model in which the function k is allowed to take values of only zero or one. In the main representation of Riella and Teper (2012), the additional psychological constraints declare an act choosable if it satisfies the lower bound on the *probability* of the states of the world in which this act under-performs relative to the default option by a certain margin. The choice

[7]A notable exception to that is Sagi (2006).

[8]At the level of setup, he studies preferences instead of choice functions; in addition, choice in his model is discontinuous.

procedure implied by this representation clearly captures a different type of behavior than the trait of mistake aversion that is studied in this paper: Indeed, it does not address the decision maker's concerns that he might be making the mistake of using a probability distribution that is slightly off — on the contrary, it puts even more emphasis on using the right probability distribution because it enters not only his objective function, but also the additional constraints.

Robust decision making and ambiguity The objective of the theory of ambiguity is to model an agent who is not sure of the probability law governing the uncertainty; ambiguity aversion, then, results in decision processes that are robust to misspecifications of the prior.

Gilboa and Schmeidler's (1989) seminal work develops a model of maxmin preferences, in which the agent considers a collection of plausible probability distributions; however, he does not discriminate among them and evaluates each of his actions using the one that represents the worst-case scenario. The subsequent works of Maccheroni, Marinacci, and Rustichini (2006) and Chateauneuf and Faro (2009) propose models of variational and confidence preferences, respectively, which can accommodate broader types of choice behavior in terms of the type of violations of the Independence axiom that they allow.

The idea of robust decision making is applied to studying macroeconomic problems in a series of papers starting with Hansen and Sargent (2001). Their constrained preferences model, however, is much closer to the maxmin model than to mine, as they impose constraints on the set of the agent's beliefs rather than on his possible courses of action.

In a separate branch of the literature, Mihm (2010) proposes a modification of the maxmin model to include an anchor — an act that serves as a reference point for evaluating all other uncertain prospects. Then, the evaluation of the anchor stays the same under all the probability distributions that the agent considers, and, hence, the anchor becomes an unambiguous prospect. In turn, the evaluations of constant acts may vary under different probability distributions, and such acts become potentially ambiguous.

The choice procedure proposed in this paper is also related to Hodges and Lehmann's (1952) restricted Bayesian solution in statistical decision theory. In comparison with that procedure, my decision maker is generally not concerned about losses for each realization of uncertainty, but about expected losses under alternative probabilistic models. In addition, the admissibility constraints in my model have a different structure: They depend on the exogenous "safe" option and aim to guarantee a particular *fraction* of the payoff guaranteed by that safe option.

3 An Application

In this section, I illustrate my model by computing the demand for and price of a risky asset in a general equilibrium setting.

Consider a one-period economy with two agents and two assets — 1) a security whose payoff is distributed log-normally with the parameters (μ, σ^2); and 2) cash. Further, suppose that Agent B has expected utility preferences with CRRA utility function with the risk aversion of $\gamma > 1$ and has correct beliefs about the distribution of the payoffs of the risky asset; at the same time, Agent A conforms to the mistake-aversion model with the same utility function, the same belief, and the tolerance function k that takes the value of k_0 on the distribution $\ln N(\mu - \beta\sigma, \sigma^2)$ — that is, the agent considers the possibility that the expected return of the risky asset may be lower. I also assume that, from Agent A's point of view, the safe option is to hold only cash. Finally, let (w_r^A, w_c^A) and (w_r^B, w_c^B) denote the endowments of the agents — the risky asset and cash, respectively — and let p denote the price of the risky asset (in units of cash).

Then, the equilibrium in this economy is described by the following conditions:

$$
\begin{aligned}
z^A = & \operatorname*{Arg\,max}_{z \in [0, w_r^A + w_c^A/p]} \int u(ze^t + w_c^A + p(w_r^A - z)) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \,\mathrm{d}t \\
& \text{s.t. } k_0 \int u(ze^t + w_c^A + p(w_r^A - z)) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu+\beta\sigma)^2}{2\sigma^2}} \,\mathrm{d}t \geq u(w_c^A + pw_r^A), \\
z^B = & \operatorname*{Arg\,max}_{z \in [0, w_r^B + w_c^B/p]} \int u(ze^t + w_c^B + p(w_r^B - z)) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \,\mathrm{d}t, \\
z^A + z^B = & \; w_r^A + w_r^B.
\end{aligned}
$$

Although Agent A's preferences are described through a choice correspondence rather than through a binary relation, his demand is, nevertheless, a well-defined and continuous function of price, and the equilibrium exists and is unique for nondegenerate values of the parameters.

As a technical remark, note that, in this optimization problem, the multiplier k_0 appears on the left-hand side of the inequality instead of the right-hand side, as in the Introduction. The reason for this is that the assumptions of this exercise do not exactly match the earlier description of my model: There, I assumed that the decision maker's utility function is bounded from below by zero; here, however, the CRRA utility function $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ is bounded by zero from *above* if the risk-aversion parameter γ is greater than one. The negative values in the inequalities require changing the way the tolerance parameter enters the expression.

The typical shape of Agent A's demand — z^A as a function of p — is depicted in Figure 1. As the figure shows, the curves representing the demands of the mistake-averse and expected utility agents diverge for intermediate values of the price p, and coincide when p is either very low or very high. Indeed, there exists a sufficiently low price p_l such that holding the risky asset dominates holding cash even under the alternative scenario (mean parameter of the distribution equal to $\mu - \beta\sigma$), and Agent A chooses to hold

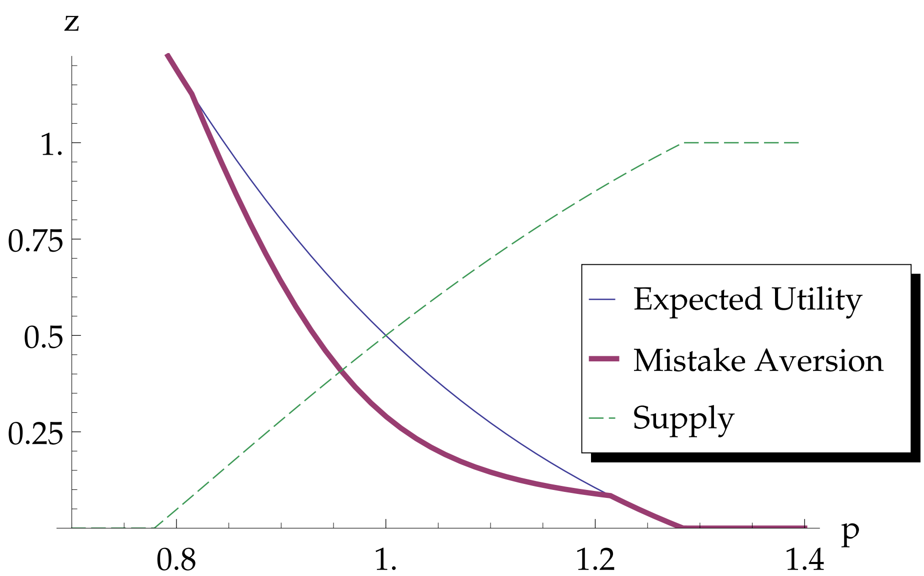

Figure 1.— Demand for a risky asset

The Mistake Aversion line depicts Agent A's demand, $z^A(p)$. The Expected Utility line is the demand of Agent A if he, instead, has expected utility preferences with the same risk-aversion parameter. The picture also shows the supply of the asset in the market from Agent A's perspective, $w_r^A + w_r^B - z^B(p)$. The values of the parameters are $\mu = \frac{1}{8}$, $\sigma = \frac{1}{2}$, $\gamma = 2$, $k_0 = 0.98$, $\beta = \frac{1}{2}$, $w_c^A = w_r^B = 1$, $w_r^A = w_c^B = 0$.

zero cash for both types of his preferences.[9] Similarly, there exists a sufficiently high price p_h such that even the expected utility agent chooses not to hold any risky asset. Since k_0 is strictly less than one, the demands of the mistake-averse and expected utility agents also coincide for prices slightly lower than p_h.

Next, Figure 2 illustrates the equilibrium expected returns of the risky asset. It shows the value of $\mathbb{E}[x/p]$, where $x \sim \ln N(\frac{1}{2}\sigma^2, \sigma^2)$ and p is the

[9]Note, also, that, in this example, the agent never shorts cash or the risky asset because doing so would imply negative consumption in some states of the world, which gives him infinitely negative utility.

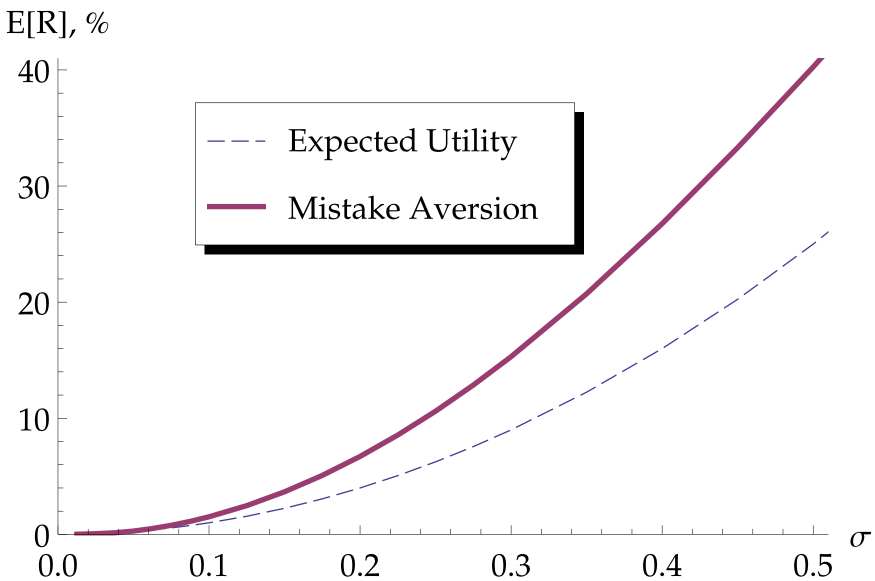

Figure 2.— Equilibrium returns of the risky asset

The Mistake Aversion line depicts, on the logarithmic scale, the expected returns of the risky asset in the economy where Agent A has mistake-averse preferences. The Expected Utility line represents the economy in which Agent A has expected utility preferences instead. The values of the parameters are $\gamma = 2$, $k_0 = 0.98$, $\beta = 1$, $w_c^A = w_r^B = 1$, $w_r^A = w_c^B = 0$.

equilibrium price, as a function of the volatility parameter σ. The two curves on the figure represent the economy described above and, for comparison, a similar economy in which Agent A has expected utility preferences instead of mistake-averse preferences. If Agent A is mistake-averse, he is less willing to hold the risky asset, and that raises the equilibrium risk premium. For the values of the parameters used in the figure and the risky asset with $\sigma = 0.2$ (which roughly corresponds to the volatility of the annual returns of the S&P500 index), the risk premium increases from about 4% to about 7%. In the presence of mistake-averse agents, the risk premium is higher than the prediction based solely on risk aversion, and this fact can potentially contribute to a better understanding of the risk premium puzzle.

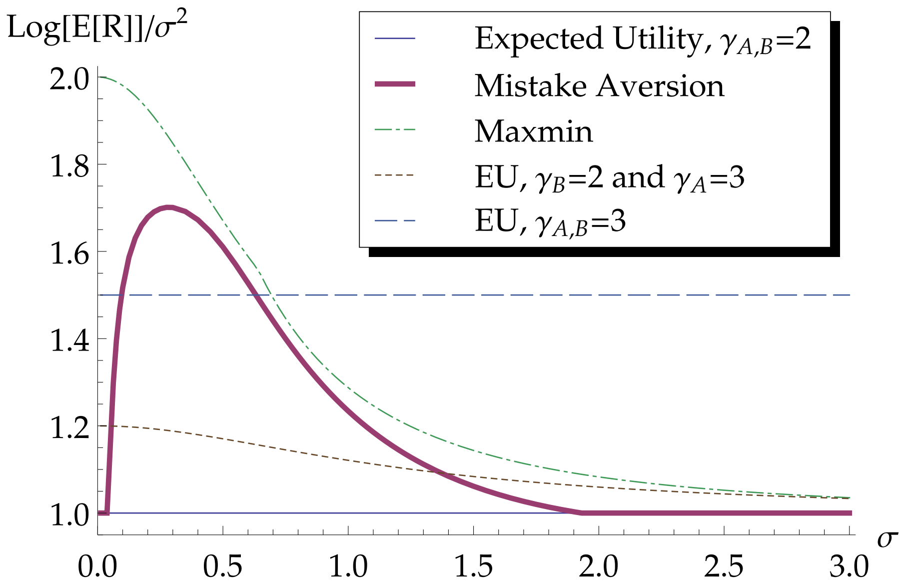

Figure 3.— Equilibrium returns of the risky asset across models

The combinations of the models used in the graph are as follows.

Agent A	Agent B
EU, $\gamma = 2$	EU, $\gamma = 2$
Mistake averse	EU, $\gamma = 2$
Maxmin	EU, $\gamma = 2$
EU, $\gamma = 3$	EU, $\gamma = 2$
EU, $\gamma = 3$	EU, $\gamma = 3$

The maxmin agent is assumed to have two priors, $\ln N(\frac{1}{2}\sigma^2, \sigma^2)$ and $\ln N(\frac{1}{2}\sigma^2 - \beta\sigma, \sigma^2)$. The values of the rest of the parameters are $k_0 = 0.98$, $\beta = 1$, $w_c^A = w_r^B = 1$, $w_r^A = w_c^B = 0$.

The effect of mistake aversion on the increase of the risk premium is, however, not uniform. As Figure 3 illustrates, one prior in the support of k raises risk premiums for the moderate levels of the volatility parameter σ; however, if the levels of volatility are either very low or very high, mistake aversion does not have that effect. The intuitive reasons for that are as follows. When σ is very low, the risky asset becomes an almost perfect substitute for cash, which implies a reduced sensitivity of the agent's expected utility to the composition of his portfolio. Then, if the tolerance parameter k_0 is less than one, the additional constraint in the agent's problem becomes not binding. At the other end of the spectrum, when σ is high, the agent's default of holding no risky assets becomes too conservative, as it ignores the potential upside. The agent, then, wants to hold at least some of the risky asset, and, as σ increases, the range of holdings that dominate the default widens. Eventually, the additional constraint in the agent's problem becomes not binding again.

The depicted shape of the returns is a feature of mistake aversion that makes it distinct from other types of cautious behavior, such as increased risk aversion or ambiguity aversion in the form of maxmin preferences. It has a number of practical implications. If the agents' risk aversion is estimated using the data on prices and holdings of assets or on financial contracts with the low volatility of payoffs, then the equilibrium risk premium of medium-volatility assets will be higher than the predictions of the standard expected utility model. At the same time, in high-volatility environments, the agents are willing to take more risk than can be expected solely on the basis of the risk aversion that they manifest in medium-volatility environments. The latter observation can potentially contribute to the discussion of market participants' willingness to take (and, conversely, insure against) catastrophic risks. The steeper slope of the curve in Figure 3 in the low- to medium-volatility range also implies that mistake aversion leads to sensitivity of the volatility of asset prices to the volatility of payoffs that is higher than in other models.

4 Axiomatic Foundations of the Model

4.1 Setup

This part of the paper examines the axiomatic foundations of the present model of mistake aversion.

To start, a formal description of the framework is in order. Let Ω be a nonempty set of the states of the world, which is assumed to be finite, and X be the space of payoffs, which is assumed to be convex.[10] The objects that the agent is evaluating are *acts* — profiles of state-contingent payoffs, represented by functions from Ω to X. The set of all acts is denoted by $\mathcal{F}$. The agent's problem is to choose his preferred acts from a finite set S of acts given a safe (or default) option d that is an element of S. His choices are described by a *choice correspondence* c that maps a decision problem (S, d) into a set $c(S, d)$ that consists of all acts in S that are equally attractive (among themselves) and strictly preferred to the rest of S.[11] The collection of all admissible decision problems is denoted by $\mathcal{C}$.[12]

4.2 Axioms

As the next step, I describe the axioms that are consistent with the objective of modeling mistake-averse choice and that, subsequently, will be proven to characterize the proposed choice procedure.

The first axiom is the standard basic postulate of choice theory: The set of chosen alternatives is always nonempty and is contained in the set of offered alternatives.

[10] More precisely, X is a convex subset of a metrizable topological vector space.

[11] As is frequent in choice theory, it is assumed that he is not forced to break ties and can choose more than one alternative if his choice set contains some options between which he is truly indifferent.

[12] Formally, $\mathcal{C} := \{(S, d) \mid S \subset \mathcal{F}, 0 < |S| < \infty, d \in S\}$.

Axiom A1 (Choice Correspondence)**.** For all $(S,d) \in \mathcal{C}$, $c(S,d) \neq \varnothing$ and $c(S,d) \subseteq S$.

Then, it is assumed that the space of outcomes X has the worst element.

Axiom A2 (Worst Element)**.** There exists $x_* \in X$ such that, for any $x \in X$, $x \in c(\{x, x_*\}, x_*)$.

Note that this axiom (and a few subsequent ones) asserts a preference between two alternatives by postulating that $x \in c(\{x, x_*\}, x_*)$, rather than $x \in c(\{x, x_*\}, x)$ — that is, via choice from a set that contains the two alternatives when the default option is taken to be the worse of them. This way of stating the axiom makes it weaker (less demanding): Overall, making an alternative the default can only increase its attractiveness, in the sense that if some alternative g is not chosen from a set S when it is the default, then it can never be chosen from S if some other alternative in S is selected to be the default.

The next axiom adapts the classical Weak Axiom of Revealed Preference to the current setting of choice with default options.

Axiom A3 (WARP)**.** For all $(S,d),(T,d) \in \mathcal{C}$ and $f,g \in \mathcal{F}$,

$$f,g \in S \cap T,\ f \in c(S,d),\ g \in c(T,d) \quad \Rightarrow \quad f \in c(T,d).$$

Generally, the Weak Axiom of Revealed Preference postulates the independence of the choice from irrelevant alternatives — the alternatives that are inferior to the ones that are chosen and, in the current setting, that are also distinct from the default. In particular, the axiom implies that

(a) if (S,d) and (T,d) are decision problems such that $T \subset S$, $f \in c(S,d)$, and $f \in T$, then it must be that $f \in c(T,d)$; and

(b) if (S,d) and (T,d) are decision problems such that $T \subset S$, $\{f,g\} \subseteq c(T,d)$, and $g \in c(S,d)$, then it must be that $f \in c(S,d)$.

In the standard theory, WARP is key to proving the transitivity of preferences inferred from choices in doubleton sets.

Next, it is assumed that choice follows the principle of monotonicity: Suppose that some alternative f is chosen in some decision problem, and g dominates f statewise (in the sense that the payoff $g(\omega)$, regardless of the state, is preferred to the certain payoff $f(\omega)$ for all $\omega \in \Omega$). Then, g must also be chosen from the same set if it becomes available.

Axiom A4 (Monotonicity)**.** For all $(S, d) \in \mathcal{C}$ and $f, g \in \mathcal{F}$,

$$\left.\begin{array}{l} f \in c(S \cup \{f\}, d) \\ g(\omega) \in c(\{g(\omega), f(\omega), x_*\}, x_*) \ \forall_{\omega \in \Omega} \end{array}\right\} \quad \Rightarrow \quad g \in c(S \cup \{g\}, d).$$

A similar monotonicity condition is standard in the theory of preferences over uncertainty.

The next axiom is a version of the classical Independence condition, which is key in the theory of expected utility preferences. Here, however, its applicability is restricted to acts that are not "ruled out" (i.e., are choosable) in the presence of the default option.

Axiom A5 (Qualified Independence)**.** For all $f, g, h, d \in \mathcal{F}$ such that $f \in c(\{f, d\}, d)$, $g \in c(\{g, d\}, d)$, and $h \in c(\{h, d\}, d)$, and all $\alpha \in (0, 1]$, we have

$$\begin{aligned} f \in c(\{f, g, d\}, d) \quad \Leftrightarrow \quad \alpha f + (1 - \alpha) h \in c(\{\alpha f + (1 - \alpha) h,& \\ \alpha g + (1 - \alpha) h, d\}, d).& \end{aligned}$$

Furthermore, it is assumed that the decision maker's understanding of mistakes is relative and invariant to changes in scale. This assumption may be viewed as parallel to assuming constant relative risk aversion in the standard theory of choice under risk. Formally, the assumption is stated as follows.

Axiom A6 (Scale Invariance)**.** For all $(S,d) \in \mathcal{C}$, $f \in \mathcal{F}$, and $\alpha \in (0,1]$,

$$f \in c(S,d) \quad \Leftrightarrow \quad \alpha f + (1-\alpha)x_* \in c(\alpha S + (1-\alpha)x_*, \alpha d + (1-\alpha)x_*).$$

In the literature on preferences over uncertainty, this postulate is similar to the Worst Independence axiom of Chateauneuf and Faro (2009). Although Scale Invariance can be viewed as an independence-like condition, it has a much more limited scope than Qualified Independence, in that it considers only mixtures with one particular element, x_*. Note, also, that the axiom is not redundant: x_* cannot play the role of h in the statement of the Qualified Independence axiom because, generally, $x_* \notin c(\{x_*, d\}, d)$.

Now, I introduce two properties that are novel in my model.

The first of the two axioms can be described as follows: Suppose that an option f is chosen over g when g is the default. Then, a mixture of f and h' should be better than a similar mixture of g and h if the options h' and h are such that h' is an improvement over h (in the sense that h' is chosen over h when h is the default).

Axiom A7 (Comparative Improvement)**.** For all $f, g, h', h \in \mathcal{F}$, and $\alpha \in [0,1]$,

$$\left.\begin{array}{l} f \in c(\{f,g\},g) \\ h' \in c(\{h',h\},h) \end{array}\right\} \quad \Rightarrow \quad \begin{aligned} \alpha f + (1-\alpha)h' \in c(\{&\alpha f + (1-\alpha)h', \\ &\alpha g + (1-\alpha)h\}, \\ &\alpha g + (1-\alpha)h). \end{aligned}$$

Generally, this axiom (as well as the next one) describes situations in which a particular choice with one default option may serve as a "justification" in other choice situations with different defaults and, therefore, restricts the types of choices that can be viewed as "mistakes."

From this viewpoint, the Comparative Improvement axiom considers the situation in which some alternative is prefered to the default (which, per force, means that choosing it is not considered a mistake), and the agent is asked to choose between the alternative and the default after they are modified (by mixing them with other alternatives) in such a way that the modification applied to the non-default alternative is preferred to the one applied to the default. Then, the axiom asserts that choosing the non-default alternative over the default in this new choice problem is perfectly justified.

The second axiom has a similar flavor.

Axiom A8 (Default Option Mirroring)**.** For all $f, d, h', h \in \mathcal{F}$ and $\alpha \in [0,1]$,

$$\begin{cases} \alpha f + (1-\alpha)h \in c(\{\alpha f + (1-\alpha)h, \alpha d + (1-\alpha)h'\}, \alpha d + (1-\alpha)h') \\ \alpha f + (1-\alpha)h' \in c(\{\alpha f + (1-\alpha)h', \alpha d + (1-\alpha)h'\}, \alpha d + (1-\alpha)h') \end{cases}$$

implies

$$\alpha f + (1-\alpha)h \in c(\{\alpha f + (1-\alpha)h, \alpha d + (1-\alpha)h\}, \alpha d + (1-\alpha)h).$$

One distinction between the Comparative Improvement and the Default Option Mirroring axioms is that the latter axiom considers changes in acts that are already mixed, whereas the former introduces a mixture only in the consequent.

To illustrate the content of this axiom, suppose that the decision maker prefers portfolio A to default portfolio D, and that A contains a fraction $1-\alpha$ of the shares of Apple, Inc. Suppose that he also prefers portfolio B — which is identical to A except that the shares of Apple, Inc., are replaced with shares of BP, plc. — to portfolio D. The axiom postulates that the decision maker, then, should also prefer A to a new default portfolio C that is obtained from D by replacing the shares of BP with shares of Apple in an amount that constitutes a $1-\alpha$ fraction of the portfolio — a change that is symmetric to the one that led from A to B. Note, also, that, in the standard expected utility theory, observing the fact that the decision maker prefers B over D would be sufficient to conclude that he should prefer A to C.

Overall, the above two axioms embody the model's objective to treat default options as reference points for the decision maker's analysis of which choices are prudent and which may turn out to be mistakes. Moreover, these axioms ensure that the default options do not have any additional effects beyond that.

The penultimate axiom in this list imposes a technical condition of the continuity of choices with respect to the changes in the choice problem.

Recall that the space of outcomes X is assumed to be endowed with a metric that I denote by $\mathfrak{d}$. Then, the space of acts $\mathcal{F}$ is assumed to be endowed with the sup-norm: $\|f-g\| := \max_{\omega\in\Omega} \mathfrak{d}(f(\omega), g(\omega))$, and a sequence $(f_n)_{n=1}^{\infty}$ converges to $f \in \mathcal{F}$ if $\lim_{n\to\infty} \|f_n - f\| = 0$.

The continuity assumption is as follows.

Axiom A9 (Continuity)**.** For all $f, g, d \in \mathcal{F}$ and sequences $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$, $(d_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $f_n \to f$, $g_n \to g$, and $d_n \to d$ as $n \to \infty$, if $f_n \in c(\{f_n, g_n, d_n\}, d_n)$ and $g_n \in c(\{g_n, d_n\})$ for all $n \in \mathbb{N}$, then $f \in c(\{f, g, d\}, d)$.

Finally, it is assumed that the agent is not totally indifferent among all alternatives.

Axiom A10 (Nondegeneracy)**.** There exists $y \in X$ such that $c(\{y, x_*\}, x_*) = \{y\}$.

4.3 Representation theorem

After introducing the axioms, I can now state the paper's main result, which establishes an equivalence between the behavioral traits captured by these axioms and the choice procedure that was introduced earlier.

Theorem 1. *A correspondence $c : \mathcal{C} \rightrightarrows \mathcal{F}$ satisfies axioms (A1)–(A10) if and only if there exists a nonconstant, continuous, and affine function $u : X \to \mathbb{R}$*

such that $\min u(X) = 0$*, a prior* $p \in \Delta(\Omega)$*, and a function* $k : \Delta(\Omega) \to [0,1]$ *such that, for all* $(S,d) \in \mathcal{C}$*,*

$$c(S,d) = \operatorname*{Arg\,max}_{f \in S} \sum_{\omega \in \Omega} u(f(\omega))\, p(\omega)$$
$$\text{s.t.} \sum_{\omega \in \Omega} u(f(\omega))\, q(\omega) \geq k(q) \sum_{\omega \in \Omega} u(d(\omega))\, q(\omega) \text{ for all } q \in \Delta(\Omega). \tag{1}$$

This theorem shows that, in fact, axioms (A1)–(A10) together impose a very strong structure on the decision maker's choice procedure:

1. The deviations from the standard expected utility maximization take the form of additional mental constraints.
2. These constraints can be assumed to be affine in the evaluated alternative f and the default option d and may be formulated in terms of their expected utilities.
3. Moreover, in each constraint, the expected utilities of f and d are computed using the same probability distribution.
4. The threshold for the expected utility of f is computed from the expected utility of d using a multiplicative factor.

Note that, in comparison to the functional form discussed in the introduction, the above theorem does not explicitly mention the set $\mathcal{M}$ of alternative probabilistic scenarios. Nevertheless, this set is present in the theorem implicitly as the support of k — the collection of probability distributions for which k takes a positive value.

In the rest of the paper, I will use the following terminology.

Definition. A choice correspondence $c : \mathcal{C} \rightrightarrows \mathcal{F}$ is said to be *guaranteed expected utility* if it admits a representation via a tuple (u,p,k) as in Theorem 1.

4.4 Guaranteed Expected Utility Preferences and Robust Decision Making

As was discussed in the introduction, the functional form (1) of the choice correspondence captures the agent's mistake aversion by eliminating the alternatives that may turn out to be mistakes. The potential mistakes here are understood as alternatives that lead to losses relative to the obvious safe choice under one of the scenarios (represented by a probability measure over Ω) that the decision maker views as possible. The name of the choice correspondence, "guaranteed expected utility," emphasizes that, in this model, the decision maker's concerns about making mistakes are solved in a way that gives him a guaranteed level of expected utility regardless of which of the possible probability distributions is true.

The crucial "parameter" of this model — the function k — serves a dual role here. First, it reflects the decision maker's *tolerance* to losses: If k decreases, he becomes less concerned about the possible size of his losses if his primary prior p turns out to be wrong. From this perspective, the level of k represents the tradeoff: When k is low, it allows the decision maker to get closer to the action that maximizes his utility under p; when k is high, it provides him with a higher level of utility that is guaranteed for all possible probability distributions. Second, k also embodies the decision maker's opinion about the *relevance* of various probability distributions: At a fixed q, the lower $k(q)$ is, the less frequently the corresponding constraint binds, and the less relevant q is for the decision process. Given that the utility function u takes only nonnegative values, the extreme case of $k(q) = 0$ means that the constraint for that q is never binding, and, thus, q is not relevant.

The function k may take various shapes. In particular, as illustrated in the next two examples, k can be chosen to express the decision maker's concern that the true probability distribution governing the uncertainty may not be exactly his primary prior, while he believes that it should still be in the proximity.

Example 1. Let function $k : \Delta(\Omega) \to [0,1]$ be defined as

$$k(q) = \begin{cases} K, & \text{if } \rho(q,p) \le \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

where $K \in (0,1]$ and $\varepsilon \ge 0$ are constants and ρ is a distance function between two probability distributions, such as relative entropy (or Kullback-Leibler distance), defined as

$$\rho(q,p) = \begin{cases} \sum\limits_{\substack{\omega \in \Omega: \\ q(\omega) > 0}} \log \dfrac{q(\omega)}{p(\omega)} q(\omega), & \text{if } q(\omega) = 0 \text{ for all } \omega \in \Omega \text{ such that } p(\omega) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this specification of k, ε determines the size of the set of probability distributions that the decision maker considers possible. All priors in this set are equally relevant for robustness analysis, and the constant K captures his overall tolerance to losses should his primary prior turn out to be wrong.

At the extreme, if $\varepsilon = +\infty$, the decision maker views all priors as possible and takes a very conservative stand by requiring a choosable action to perform relatively well under any of them. At the other extreme, if $\varepsilon = 0$, the model collapses to the standard expected utility maximization (and, as will be elaborated in the next section, the value of K in this case becomes irrelevant).

Example 2. Let function $k : \Delta(\Omega) \to [0,1]$ be defined as

$$k(q) = \begin{cases} 1 - \frac{1}{\varepsilon}\rho(q,p), & \text{if } \rho(q,p) \le \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is a constant and ρ is a distance function. This specification has only one numeric parameter, ε, that determines the critical distance between the priors that are still considered possible and the primary prior p. The alternative probabilistic scenarios now become not equally important: Their relevance takes the maximal value of one at p and fades to zero as the distance between the prior and p increases.

Overall, the choice procedure (1) is in line with the procedures that can be found in practice. A related idea of robust choice is expressed, for example, in the famous (albeit controversial) book *Fooled by Randomness* by Nassim Nicholas Taleb, which is one of very few books on investing featured in *Fortune*'s list of "The Smartest Books We Know." Writing about financial decisions based on probabilities that are not objective but estimated from the data or inferred from previous experience, Taleb states: "I will use statistics and inductive methods to make aggressive bets, but I will not use them to manage my risk and exposure. Surprisingly, all the surviving traders I know seem to have done the same. They trade on ideas based on some observation ... but ... they make sure that the costs of being wrong are limited[.]" The procedure studied in this paper is also consistent with formal risk management in financial institutions, which maximize profits of their investing activities subject to risk limits that are imposed either internally or by the regulator. While not all risk limits take the form of a threshold on possible expected losses, many of them, such as stress loss, Credit Spread Basis Point Value, and Credit Spread Widening 10%, are consistent with (1).[13]

[13]The stress loss metric shows the potential mark-to-market loss of a portfolio under certain hypothetical abnormal market conditions, such as an oil crisis, a repetition of the Asian'97 crisis, and others. Credit Spread Basis Point Value is the loss (or gain) under a hypothetical deterioration of the credit markets in which all credit spreads (the cost of insurance against default) increase by one basis point in absolute terms. Credit Spread

4.5 Uniqueness of the Representation

While Theorem 1 establishes that, observing the agent's choices, one can find a utility function, a prior, and a tolerance function that represent his choice procedure, it is also desirable to understand the extent to which these objects can be identified uniquely. Answering this question is important not only for interpreting the model, but also for doing the comparative statics, calibrating the model in applications, and so on.

The uniqueness properties of the representation in Theorem 1 resemble the properties of a number of models in the literature on ambiguity, such as Chateauneuf and Faro's (2009), Maccheroni et al. (2006), and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011): Although identical choice behavior can be generated by more than one tuple of the object of the representation, in the set of the "equivalent" tuples, there is a salient one.

As the next theorem shows, the salient representation in my model is the one that has the maximal k: In this case, the additional mental constraints become binding for all $q \in \Delta(\Omega)$ such that $k(q) > 0$.

Theorem 2. *Suppose that a choice correspondence $c : \mathcal{C} \rightrightarrows \mathcal{F}$ satisfies axioms (A1)–(A10), and (u, p, k) is its representation as in 1. Then:*

- *(i) If (u', p', k') is another representation of c, then $u' = \beta u$ for some $\beta > 0$, $p' = p$, and $(u, p, \max(k, k'))$ is also a representation of c.*[14]
- *(ii) There exists a (unique) pointwise-maximal $k^* : \Delta(\Omega) \to [0, 1]$ such that (u, p, k^*) is a representation of c.*

Since my model can accommodate the standard expected utility prefer-

Widening 10% is a similar loss or gain if all credit spreads increase in relative terms by 10% of their current values.

[14] The maximum operation is taken pointwise: $\max(k, k')(q) = \max(k(q), k'(q))$ for all $q \in \Delta(\Omega)$.

ences by letting $k(q) = 0$ for all $q \in \Delta(\Omega)$, it may also be useful to re-state the uniqueness claim for this special form of preferences.

Observation 3. *Suppose that $c : \mathcal{C} \rightrightarrows \mathcal{F}$ is a choice correspondence such that there exists a nonconstant, continuous, and affine function $u : X \to \mathbb{R}_+$ such that $\min u(X) = 0$, and $p \in \Delta(\Omega)$ such that*

$$c(S,d) = \operatorname*{Arg\,max}_{f \in S} \sum_{\omega \in \Omega} u(f(\omega))\, p(\omega).$$

Then, (u', p', k') is a representation of c as in (1) if and only if $u' = \beta u$ for some $\beta > 0$, $p' = p$, and $k'(q) = 0$ for all $q \in \Delta(\Omega) \backslash \{p\}$.

The central part of this statement is that, in the representation of expected utility preferences, the value of the tolerance function $k(q)$ is unique (and is equal to zero) for all q other than p.

This observation also highlights a property of the guaranteed expected utility representation that holds for any choice correspondence in the model: The value of the tolerance function at the prior p is irrelevant for the choice behavior. Indeed, in any decision problem $(S,d) \in \mathcal{C}$, if $\sum_{\omega \in \Omega} u(f(\omega))\, p(\omega) < \sum_{\omega \in \Omega} u(d(\omega))\, p(\omega)$, then f is never chosen in (S,d) regardless of the value of $k(p)$; conversely, if f is chosen, then it must be that $\sum_{\omega \in \Omega} u(f(\omega))\, p(\omega) \geq \sum_{\omega \in \Omega} u(d(\omega))\, p(\omega)$, and the constraint for $q = p$ is satisfied for any value of $k(p) \in [0,1]$. Furthermore, this observation also implies that the maximal tolerance k^* at p is always one.

4.6 Comparative Statics

To complete the model, it is necessary to develop a way to compare decision makers' attitudes towards mistakes and characterize them in terms of the representation for the purpose of comparative statics.

The key concept in this analysis is the behavioral notion of one decision maker being more mistake-averse than another.

Definition 1. A choice correspondence $c_2 : \mathcal{C} \rightrightarrows \mathcal{F}$ is said to be *more mistake-averse* than $c_1 : \mathcal{C} \rightrightarrows \mathcal{F}$ if, for all $(S, d) \in \mathcal{C}$, $\{d\} = c_1(S, d)$ implies $\{d\} = c_2(S, d)$.

In words, fix a decision problem and suppose that one decision maker strictly prefers to keep the default option in that problem. Then, a decision maker who is more mistake-averse than the first should be even more compelled to keep the default.

The next proposition provides a characterization of that property in terms of the representation.

Proposition 4. *Suppose that $c_1, c_2 : \mathcal{C} \rightrightarrows \mathcal{F}$ are guaranteed expected utility choice correspondences, and (u_1, p_1, k_1^*) and (u_2, p_2, k_2^*) are their representations with the maximal k_1^* and k_2^*. Then, c_2 is more mistake-averse than c_1 if and only if $u_2 = \beta u_1$ for some $\beta > 0$ and $k_2^*(q) \geq k_1^*(q)$ for all $q \in \Delta(\Omega)$.*

This proposition establishes that one decision maker is more mistake-averse than another if and only if he has less tolerance (and, hence, a higher acceptance threshold) for losses under all alternative probabilistic scenarios. Note that this comparative notion — one agent being more mistake-averse than another — does not impose any direct restrictions on how their beliefs — p_1 and p_2 — must be related; indirectly, the characterization of the proposition implies that $k_2^*(p_1) = 1$.

While Proposition 4 provides a characterization in terms of the maximal tolerance functions, it may also be useful to state a partial result about the effect of changing the tolerance function in an arbitrary (not necessarily maximal) representation.

Proposition 5. *Suppose that $c_1, c_2 : \mathcal{C} \rightrightarrows \mathcal{F}$ are guaranteed expected utility choice correspondences, and (u, p, k_1) and (u, p, k_2) are their representa-*

tions. Then, $k_2(q) \geq k_1(q)$ *for all* $q \in \Delta(\Omega)$ *implies that* c_2 *is more mistake-averse than* c_1*.*

Considering, again, the special case of the standard expected utility preferences, we have the following trivial corollary.

Corollary 6. *For any guaranteed expected utility choice correspondence* c*, there is a standard expected utility choice correspondence* c^0 *such that* c *is more mistake averse than* c^0*.*

This means that, indeed, choice correspondences from the guaranteed expected utility class can rightfully be called mistake-averse, and that the standard preferences may be thought of as "mistake-neutral."

Appendix

A Representation Through Binary Relations

In this section, I develop an intermediate representation of choice correspondences that satisfy axioms (A1)–(A10) via a pair of binary relations.

Before proceeding to the results, I formally introduce a number of properties of an arbitrary binary relation $\trianglerighteq$ on $\mathcal{F}$.

Reflexivity For all $f \in \mathcal{F}$, $f \trianglerighteq f$.

Transitivity For all $f, g, h \in \mathcal{F}$, $f \trianglerighteq g$ and $g \trianglerighteq h$ imply $f \trianglerighteq h$.

Completeness For all $f, g \in \mathcal{F}$, either $f \trianglerighteq g$ or $g \trianglerighteq f$.

Monotonicity For all $f, g \in \mathcal{F}$ such that $f(\omega) \trianglerighteq g(\omega)$ for all $\omega \in \Omega$, we have $f \trianglerighteq g$.

Independence For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1]$, $f \trianglerighteq g \Leftrightarrow \alpha f + (1 - \alpha)h \trianglerighteq \alpha g + (1 - \alpha)h$.

Continuity For all $f, g \in \mathcal{F}$ and sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ such that $f_n \trianglerighteq g_n$ for all $n \in \mathbb{N}$ and $f_n \to f$, $g_n \to g$ as $n \to \infty$, we have $f \trianglerighteq g$.

Nondegeneracy There exist $f, g \in \mathcal{F}$ such that $f \triangleright g$.

Worst Element There exists $x_* \in X$ such that $x \trianglerighteq x_*$ for all $x \in X$.

Scale Invariance For all $f, g \in \mathcal{F}$ and $\alpha \in (0, 1]$, $f \trianglerighteq g \Leftrightarrow \alpha f + (1 - \alpha)x_* \trianglerighteq \alpha g + (1 - \alpha)x_*$.

Comparative Improvement For all $f, g, h', h \in \mathcal{F}$ and $\alpha \in [0, 1]$, $f \trianglerighteq g$ and $h' \trianglerighteq h$ implies $\alpha f + (1 - \alpha)h' \trianglerighteq \alpha g + (1 - \alpha)h$.

Mirroring For all $f, g, h', h \in \mathcal{F}$ and $\alpha \in [0, 1]$,

$$\begin{cases} \alpha f + (1 - \alpha)h \trianglerighteq \alpha g + (1 - \alpha)h' \\ \alpha f + (1 - \alpha)h' \trianglerighteq \alpha g + (1 - \alpha)h' \end{cases}$$

implies

$$\alpha f + (1 - \alpha)h \trianglerighteq \alpha g + (1 - \alpha)h).$$

Next, I state and prove a number of lemmas that will be used to prove the main result of this appendix.

Lemma 7. *Suppose that $c : \mathcal{C} \rightrightarrows \mathcal{F}$ satisfies Choice Correspondence, Worst Element, and Monotonicity axioms. Then, for any $f \in \mathcal{F}$, $f \in c(\{f, x_*\}, x_*)$.*

Proof. Observe that $f(\omega) \in c(\{f(\omega), x_*\}, x_*)$ for all $\omega \in \Omega$ by the Worst Element axiom. Since $x_* \in c(\{x_*\}, x_*)$ by the Choice Correspondence axiom, we obtain $f \in c(\{f, x_*\}, x_*)$ by the Monotonicity axiom. $\square$

Lemma 8. *Suppose that $c : \mathcal{C} \rightrightarrows \mathcal{F}$ is a correspondence that satisfies axioms (A1)–(A10), and suppose that $f, g, d \in \mathcal{F}$ are such that $f \in c(\{f, d\}, d)$ and $g \in c(\{g, d\}, d)$. Then,*

$$f \in c(\{f, g, d\}, d) \quad \Leftrightarrow \quad f \in c(\{f, g, x_*\}, x_*). \tag{2}$$

Proof. "$\Rightarrow$" part. Suppose that $f \in c(\{f, g, d\}, d)$. Fix an arbitrary $t \in (0,1)$, and let $d' := td + (1-t)x_*$, $f' := tf + (1-t)x_*$, $g' := tg + (1-t)x_*$. By the Scale Invariance axiom, we have $f' \in c(\{f', d'\}, d')$, $g' \in c(\{g', d'\}, d')$, and $f' \in c(\{f', g', d'\}, d')$.

Next, let $f^0 := \frac{1}{1+t} f' + \frac{t}{1+t} d$ and $g^0 := \frac{1}{1+t} g' + \frac{t}{1+t} d$. Observe that $d \in c(\{d, d'\}, d')$ by the Comparative Improvement axiom: The conditions of the axiom hold because $d \in c(\{d\}, d)$ by the Choice Correspondence axiom, and $d \in c(\{d, x_*\}, x_*)$ by Lemma 7. Then, by the Qualified Independence axiom, we have $f^0 \in c(\{f^0, g^0, d'\}, d')$.

Observe, also, that $f^0 = \frac{t}{1+t} f + \frac{1}{1+t} d'$ and $g^0 = \frac{t}{1+t} g + \frac{1}{1+t} d'$. In addition, $f \in c(\{f, d'\}, d')$ and $g \in c(\{g, d'\}, d')$ by the Comparative Improvement axiom: The conditions of the axiom hold because $f \in c(\{f, d\}, d)$ and $g \in c(\{g, d\}, d)$ by assumption, and $f \in c(\{f, x_*\}, x_*)$ and $g \in c(\{g, x_*\}, x_*)$ by Lemma 7. Therefore, by the Qualified Independence axiom, we have $f \in c(\{f, g, d'\}, d')$.

Finally, since $f \in c(\{f, g, d'\}, d')$ holds for all $t \in (0,1)$, then, by the Continuity axiom, the same must hold in the limit as $t \to 0$, and we obtain $f \in c(\{f, g, x_*\}, x_*)$.

"$\Leftarrow$" part. Suppose that $f \in c(\{f, g, x_*\}, x_*)$ and assume, by contradiction, that $f \notin c(\{f, g, d\}, d)$.

Step 1. I claim that $\{g\} = c(\{f, g, d\}, d)$. Indeed, $c(\{f, g, d\}, d)$ is a nonempty subset of $\{f, g, d\}$ by the Choice Correspondence axiom, and $d \notin c(\{f, g, d\}, d)$: Otherwise, WARP and the fact that $f \in c(\{f, d\}, d)$ would imply that $f \in c(\{f, g, d\}, d)$, which is assumed to be not the case.

Step 2. Let x^* be an arbitrary element of X such that $c(\{x^*, x_*\}, x_*) = \{x^*\}$, and note that it exists by the Nondegeneracy axiom. For each $t \in [0, 1]$, let f'_t, f''_t, g'_t, d'_t be defined as follows:

$$f' := (1 - t)f + tx_*, \qquad f'' := (1 - t)f + tx^*,$$
$$g' := (1 - t)g + tx_*, \qquad d' := (1 - t)d + tx_*.$$

Observe that $f'_t \in c(\{f'_t, d'_t\}, d'_t)$ and $g'_t \in c(\{g'_t, d'_t\}, d'_t)$ for all $t \in (0, 1)$ by the Scale Invariance axiom, and $d'_t \in c(\{d'_t\}, d'_t)$ for all $t \in (0, 1)$ by Choice Correspondence.

Step 3. I also claim that $f''_t \in c(\{f''_t, d'_t\}, d'_t)$ for all $t \in (0, 1)$. Indeed, for all $\omega \in \Omega$ and all $t \in (0, 1)$, we have $f''_t(\omega) \in c(\{f''_t(\omega), f'_t(\omega), x_*\}, x_*)$ by the Qualified Invariance axiom: The pair $(f''_t(\omega), f'_t(\omega))$ is a mixture with $f_t(\omega)$ of the pair (x^*, x_*) that satisfies $x^* \in c(\{x^*, x_*\}, x_*)$. The claim of this step, then, follows from $f'_t \in c(\{f'_t, d'_t\}, d'_t)$ by the Monotonicity axiom.

Step 4. By the Continuity axiom, the result of Step 1 implies that it is possible to choose a sufficiently small $\tau \in (0, 1)$ such that $\{g'_\tau\} = c(\{f''_\tau, g'_\tau, d'_\tau\}, d'_\tau)$. Then, as follows from the already proven part of this Lemma, we have $g'_\tau \in c(\{f''_\tau, g'_\tau, x_*\}, x_*)$.

Step 5. Since $f \in c(\{f, g, x_*\}, x_*)$ by assumption, we also have $f'_\tau \in c(\{f'_\tau, g'_\tau, x_*\}, x_*)$ by the Scale Invariance axiom.

Step 6. Let $C = c(\{f'_\tau, f''_\tau, g'_\tau, x_*\}, x_*)$, and note that it is a nonempty subset of $\{f'_\tau, f''_\tau, g'_\tau, x_*\}$ by the Choice Correspondence axiom. Now, I prove that $f'_\tau \in C$ by using WARP repeatedly: If $x_* \in C$, then $f'_\tau \in C$ because $f'_\tau \in c(\{f'_\tau, x_*\}, x_*)$ by the Worst Element axiom. If $f''_\tau \in C$, then $g'_\tau \in C$ because $g'_\tau \in c(\{f''_\tau, g'_\tau, x_*\}, x_*)$ by Step 4. If $g'_\tau \in C$, then $f'_\tau \in C$ because $f'_\tau \in c(\{f'_\tau, g'_\tau, x_*\}, x_*)$ by Step 5. Jointly, these observations imply that C contains f'_τ in any case.

Step 7. Finally, by WARP, $f'_\tau \in C$ implies that $f'_\tau \in c(\{f'_\tau, f''_\tau, x_*\}, x_*)$. By the Qualified Independence axiom, it implies that $x_* \in c(\{x_*, x^*\}, x_*)$, which contradicts the choice of x^*. $\square$

Lemma 9. *Suppose that $c : \mathcal{C} \rightrightarrows \mathcal{F}$ is a correspondence that satisfies axioms (A1)–(A10). Then, the binary relation $\succsim^*$ defined as*

$$f \succsim^* g \quad \Leftrightarrow \quad f \in c(\{f, g, x_*\}, x_*) \tag{3}$$

satisfies Reflexivity, Transitivity, Completeness, Monotonicity, Independence, Continuity, and Nondegeneracy.

Proof. The proof of Reflexivity, Transitivity, and Completeness follows the standard argument, with the only amendment that our setting requires the default option to be present in all choice sets. For all $f \in \mathcal{F}$, $f \in c(\{f, x_*\}, x_*)$ by the Worst Element axiom, which proves Reflexivity.

To prove Transitivity, suppose that $f_1, f_2, f_3 \in \mathcal{F}$ are such that $f_1 \succsim^* f_2$ and $f_2 \succsim^* f_3$, which means that $f_1 \in c(\{f_1, f_2, x_*\}, x_*)$ and $f_2 \in c(\{f_2, f_3, x_*\}, x_*)$. Let $S := c(\{f_1, f_2, f_3, x_*\}, x_*)$. I claim that $f_1 \in S$. Indeed, if $x_* \in S$, then $f_1 \in S$ by WARP as $f_1 \in c(\{f_1, x_*\}, x_*)$ by the Worst Element axiom. Suppose that $x_* \notin S$. Then, by the Choice Correspondence axiom, S is a nonempty subset of $\{f_1, f_2, f_3\}$. Let $i \in \{1, 2, 3\}$ be the smallest number such that $f_i \in S$. If $i \neq 1$, then $f_{i-1} \in S$ by WARP because $f_{i-1} \in c(\{f_{i-1}, f_i, x_*\}, x_*)$, a contradiction to the choice of i. This completes the proof of the claim that $f_1 \in S$. Now, by WARP again, $f_1 \in c(\{f_1, f_3, x_*\}, x_*)$, which means that $f_1 \succsim^* f_3$. The Transitivity is now proven.

To prove completeness, let $f, g \in \mathcal{F}$ be arbitrary, and let $S := c(\{f, g, x_*\}, x_*)$. I claim that either $f \in S$ or $g \in S$ (or both). Indeed, if neither of them is in S, then, by the Choice Correspondence axiom, it must be that $S = \{x_*\}$; in this case, it must be also that $f \in S$ by WARP as $f \in c(\{f, x_*\}, x_*)$ by the Worst Element axiom, a contradiction. If $f \in S$, then $f \succsim^* g$, and if $g \in S$, then $g \succsim^* f$, and the Completeness is now proven.

The Monotonicity, Independence, Continuity, and Nondegeneracy properties of $\succsim^*$ follow from the corresponding axioms immediately. $\square$

Now, we are ready to prove the main result of this appendix — a representation of choice correspondences through a pair of binary relations.

Proposition 10. *Suppose $c : \mathcal{C} \rightrightarrows \mathcal{F}$ is a correspondence that satisfies axioms (A1)–(A10). Then, there exist two binary relations $\succsim^*$ and $\succsim$ on $\mathcal{F}$*

such that

$$c(S,d) = \mathrm{Max}_{\succsim^*}\{f \mid f \in S, f \succsim d\}, \tag{4}$$

and such that

(i) $\succsim^*$ *satisfies Reflexivity, Transitivity, Completeness, Monotonicity, Independence, Continuity, and Nondegeneracy;*

(ii) $\succsim$ *satisfies Reflexivity, Worst Element, Scale Invariance, Comparative Improvement, Mirroring, and Continuity;*

(iii) $\succsim^*$ *and* $\succsim$ *jointly have the following properties:*

(a) For all $f, g \in \mathcal{F}$, $f \succsim g$ *implies* $f \succsim^* g$.

(b) For all $d \in \mathcal{F}$ *and all* $f, g \in \mathcal{F}$ *such that* $g(\omega) \succsim^* f(\omega)$ *for all* $\omega \in \Omega$*, we have* $f \succsim d \Rightarrow g \succsim d$.

Proof. Let $\succsim^*$ be defined by (3), and $\succsim$ be defined as $f \succsim g \Leftrightarrow f \in c(\{f,g\}, g)$.

Let (S,d) be an arbitrary choice problem in $\mathcal{C}$, and $f \in c(S,d)$. I claim that, first, $f \succsim d$, and, second, $f \succsim^* g$ for all $g \in S$ such that $g \succsim d$. The first part of the claim, $f \succsim d$, follows immediately from WARP. Let $g \in S$ be an arbitrary element such that $g \succsim d$. Since $f \in c(S,d)$, then $f \in c(\{f,g,d\}, d)$ by WARP. By Lemma 8, $f \in c(\{f,g,x_*\}, x_*)$, which means that $f \succsim^* g$.

Conversely, suppose that $(S,d) \in \mathcal{C}$, $f \in S$, $f \succsim d$, and $f \succsim^* g$ for all $g \in S'$, where $S' := \{h \in S : h \succsim d\}$. By Lemma 8, $f \in c(\{f,g,d\}, d)$ for all $g \in S'$. We also have $f \in c(\{f,g,d\}, d)$ for all $g \in S \backslash S'$ by WARP since $g \notin c(\{g,d\}, d)$. Using WARP again, we conclude that $f \in c(S,d)$.

It remains to prove that $\succsim^*$ and $\succsim$ have the properties that were claimed in the statement of the proposition. The properties of $\succsim^*$ are established by Lemma 9. Similarly to $\succsim'$, the reflexivity of $\succsim$ follows from the Worst Element axiom. Worst Element, Scale Invariance, Comparative Improvement, and Continuity properties of $\succsim$ follow immediately from the corresponding axioms, and the Mirroring property follows from the Default Option Mirroring axiom. Finally, Claim (iii)(a) follows from Lemma 8, and Claim (iii)(b) follows immediately from the Monotonicity axiom. $\square$

B Proofs of the Results

Recall that the state space Ω is currently assumed to be finite, and this assumption is used in Lemma 15. The extension of the results to the infinite state space and infinite-dimensional space of acts is work in progress; however, for most of this appendix, the notation is already adapted to that extension.

B.1 Binary relations over utility acts

As is standard in the literature on decisions over uncertainty, it is useful to switch from working with binary relations over acts in $\mathcal{F}$ to binary relations on so-called utility acts. The possibility of doing that in the current setting is the subject of this subsection.

Notation. For any nondegenerate interval $I \subseteq \mathbb{R}$, let $\mathcal{B}(I) := I^{\Omega}$. Also, let $ba(\Omega)$ denote the set of all additive functions from 2^{Ω} to $\mathbb{R}$.

I define the following properties of an arbitrary binary relation $\succsim$ on $\mathcal{B}(I)$, where $I \subseteq \mathbb{R}$ is a nondegenerate interval containing 0.

Reflexivity For all $f \in \mathcal{B}(I)$, $f \succsim f$.

Comparative Monotonicity For all $d \in \mathcal{B}(I)$ and all $f, g \in \mathcal{B}(I)$ such that $g \geq f$ pointwise, we have $f \succsim d \Rightarrow g \succsim d$.

Scale Invariance For all $f, g \in \mathcal{B}(I)$ and all $\alpha \in (0,1]$, $f \succsim g \Leftrightarrow \alpha f \succsim \alpha g$.

Comparative Improvement For all $f, g, h', h \in \mathcal{B}(I)$ and $\alpha \in [0,1]$, $f \succsim g$ and $h' \succsim h$ implies $\alpha f + (1-\alpha)h' \succsim \alpha g + (1-\alpha)h$.

Mirroring For all $f, g, h', h \in \mathcal{B}(I)$ and $\alpha \in [0,1]$,

$$\begin{cases} \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h' \\ \alpha f + (1-\alpha)h' \succsim \alpha g + (1-\alpha)h' \end{cases}$$

implies

$$\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h).$$

Continuity For all $f, g \in \mathcal{B}(I)$ and sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ such that $f_n \succsim g_n$ for all $n \in \mathbb{N}$ and $f_n \to f$, $g_n \to g$ in the sup-norm as $n \to \infty$, we have $f \succsim g$.

Lemma 11. *Suppose that $\succsim$ is a binary relation on $\mathcal{F}$ that satisfies Reflexivity, Worst Element, Scale Invariance, Comparative Improvement, Mirroring, and Continuity, as defined in Appendix A. Suppose, also, that there exists a nonconstant, continuous, and affine function $u : X \to \mathbb{R}$ such that*

(i) For all $x, y \in X$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$;

(ii) $u(x_*) = 0$;

(iii) For all $\tilde{d} \in \mathcal{F}$ and all $\tilde{f}, \tilde{g} \in \mathcal{F}$ such that $u \circ \tilde{g} \geq u \circ \tilde{f}$ pointwise, we have $\tilde{f} \succsim \tilde{d} \Rightarrow \tilde{g} \succsim \tilde{d}$.

Then, there exists a binary relation $\succsim^\circ$ on $\mathcal{B}(I)$, where $I := u(X)$, that satisfies Reflexivity, Comparative Monotonicity, Scale Invariance, Comparative Improvement, Mirroring, and Continuity, as defined above, and such that, for all $\tilde{f}, \tilde{g} \in \mathcal{F}$,

$$\tilde{f} \succsim \tilde{g} \quad \Leftrightarrow \quad (u \circ \tilde{f}) \succsim^\circ (u \circ \tilde{g}).$$

Proof. Step 1. Let $\succsim^\circ$ be defined as $f \succsim^\circ g$ if and only if there exist $\tilde{f}, \tilde{g} \in \mathcal{F}$ such that $u \circ \tilde{f} = f$, $u \circ \tilde{g} = g$, and $\tilde{f} \succsim \tilde{g}$.

Step 2. Suppose that $\tilde{f}, \tilde{f}', \tilde{g} \in \mathcal{F}$ are such that $u \circ \tilde{f} = u \circ \tilde{f}'$ and $\tilde{f} \succsim \tilde{g}$. I claim that $\tilde{f}' \succsim \tilde{g}$. Indeed, if $u \circ \tilde{f} = u \circ \tilde{f}'$, then $\tilde{f}'(\omega) \sim \tilde{f}(\omega)$ for all $\omega \in \Omega$. Therefore, by Comparative Monotonicity, $\tilde{f}' \succsim \tilde{g}$.

Step 3. Suppose that $\tilde{f}, \tilde{g}, \tilde{g}' \in \mathcal{F}$ are such that $u \circ \tilde{g} = u \circ \tilde{g}'$ and $\tilde{f} \succsim \tilde{g}$. I claim that $\tilde{f} \succsim \tilde{g}'$.

Fix an arbitrary $n \in \mathbb{N}$ such that $n \geq 2$. I prove, first, that

$$\left(1 - \frac{1}{n}\right)\tilde{f} + \frac{1}{n}\tilde{g} \succsim \frac{n-k}{n}\tilde{g} + \frac{k}{n}\tilde{g}' \tag{5}$$

for all $k = 0, \ldots, n$. Indeed, for $k = 0$, this claim follows from Comparative Improvement since $g \succsim g$ by Reflexivity. Next, suppose, by induction, that (5) holds for some $k = 0, \ldots, n-1$. By Step 2, (5) also implies

$$\left(1 - \frac{1}{n}\right)\tilde{f} + \frac{1}{n}\tilde{g}' \succsim \frac{n-k}{n}\tilde{g} + \frac{k}{n}\tilde{g}'. \tag{6}$$

By Mirroring, (5) and (6) together imply that

$$\left(1-\frac{1}{n}\right)\tilde{f}+\frac{1}{n}\tilde{g}'\succsim\frac{n-k-1}{n}\tilde{g}+\frac{k+1}{n}\tilde{g}'.$$

By Step 2, $\tilde{g}'$ can be replaced with $\tilde{g}$ in the left-hand side of the above expression, and the induction step is complete. Hence, (5) is proven for all $k=0,\dots,n$.

Then, substituting $k=n$ into (5), we obtain

$$\left(1-\frac{1}{n}\right)\tilde{f}+\frac{1}{n}\tilde{g}'\succsim\tilde{g}'.$$

Taking $n\to\infty$ and using Continuity, this gives $\tilde{f}\succsim\tilde{g}'$.

Step 4. Combining the results of Steps 2 and 3, we conclude that if $f\succsim^{\circ}g$ for some $f,g\in\mathcal{B}(I)$ and $\tilde{f},\tilde{g}\in\mathcal{F}$ are such that $u\circ\tilde{f}=f$ and $u\circ\tilde{g}=g$, then it must be that $\tilde{f}\succsim\tilde{g}$.

Step 5. Now, I prove the Mirroring property. Suppose that $f,g,h',h\in\mathcal{B}(I)$ and $\alpha\in[0,1]$ are such that $\alpha f+(1-\alpha)h\succsim^{\circ}\alpha g+(1-\alpha)h'$ and $\alpha f+(1-\alpha)h'\succsim^{\circ}\alpha g+(1-\alpha)h'$. Choose arbitrary $\tilde{f},\tilde{g},\tilde{h}',\tilde{h}\in\mathcal{F}$ such that $u\circ\tilde{f}=f$, $u\circ\tilde{g}=g$, $u\circ\tilde{h}'=h'$, and $u\circ\tilde{h}=h$. Since u is affine, we have $u\circ\left[\alpha\tilde{f}+(1-\alpha)\tilde{h}\right]=\alpha f+(1-\alpha)h$, $u\circ\left[\alpha\tilde{g}+(1-\alpha)\tilde{h}'\right]=\alpha g+(1-\alpha)h'$, $u\circ\left[\alpha\tilde{f}+(1-\alpha)\tilde{h}'\right]=\alpha f+(1-\alpha)h'$. Since $\alpha f+(1-\alpha)h\succsim^{\circ}\alpha g+(1-\alpha)h'$ and $\alpha f+(1-\alpha)h'\succsim^{\circ}\alpha g+(1-\alpha)h'$, then, by the result of Step 4, we have $\alpha\tilde{f}+(1-\alpha)\tilde{h}\succsim\alpha\tilde{g}+(1-\alpha)\tilde{h}'$ and $\alpha\tilde{f}+(1-\alpha)\tilde{h}'\succsim\alpha\tilde{g}+(1-\alpha)\tilde{h}'$. Therefore, by Mirroring, we have $\alpha\tilde{f}+(1-\alpha)\tilde{h}\succsim\alpha\tilde{g}+(1-\alpha)\tilde{h}$, which gives $\alpha f+(1-\alpha)h\succsim^{\circ}\alpha g+(1-\alpha)h$ by the definition of $\succsim^{\circ}$ and the affinity of u.

Step 6. Now, I prove the Continuity property. Suppose that $f,g\in\mathcal{B}(I)$ and sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ in $\mathcal{B}(I)$ are such that $f_n\succsim^{\circ}g_n$ for all $n\in\mathbb{N}$, and $f_n\to f$ and $g_n\to g$ as $n\to\infty$.

Step 6a. Let

$$\begin{aligned}\underline{a}&:=\min\{\min_{\omega\in\Omega}f(\omega),\min_{\omega\in\Omega}g(\omega)\},\\ \bar{a}&:=\max\{\max_{\omega\in\Omega}f(\omega),\max_{\omega\in\Omega}g(\omega)\}.\end{aligned}$$

If $\underline{a}=0$, then let $\underline{x}=x_*$. Otherwise, let $\underline{x}\in X$ be such that $u(\underline{x})<\underline{a}$. If $\bar{a}\geq a$ for all $a\in u(X)$, then let $\bar{x}\in X$ be such that $u(\bar{x})=\bar{a}$. Otherwise, let $\bar{x}\in X$

be such that $u(\bar{x}) > \bar{a}$. Note that $u(\underline{x}) < u(\bar{x})$ because u is nonconstant. Now, let $\gamma : u(X) \to X$ be defined as follows. For $a < u(\underline{x})$ or $a > u(\bar{x})$, let $\gamma(a)$ be chosen arbitrary subject to $u(\gamma(a)) = a$. For $a \in [u(\underline{x}), u(\bar{x})]$, let $\gamma(a)$ be a mixture of $\underline{x}$ and $\bar{x}$ such that $u(\gamma(a)) = a$; that is,

$$\gamma(a) := \frac{u(\bar{x}) - a}{u(\bar{x}) - u(\underline{x})}\underline{x} + \frac{a - u(\underline{x})}{u(\bar{x}) - u(\underline{x})}\bar{x}.$$

Step 6b. Observe that, for any $a, a' \in [u(\underline{x}), u(\bar{x})]$, we have $\gamma(a) - \gamma(a') = \frac{a-a'}{u(\bar{x})-u(\underline{x})}(\bar{x} - \underline{x})$. By the construction of $\underline{x}$ and $\bar{x}$, we have $f(\omega) \in [u(\underline{x}), u(\bar{x})]$ and $g(\omega) \in [u(\underline{x}), u(\bar{x})]$ for all $\omega \in \Omega$ and, simultaneously, there exists $N \in \mathbb{N}$ such that, for all $n > N$ and all $\omega \in \Omega$, we have $f_n(\omega) \in [u(\underline{x}), u(\bar{x})]$ and $g_n(\omega) \in [u(\underline{x}), u(\bar{x})]$. Therefore, $\gamma \circ f_n \to \gamma \circ f$ and $\gamma \circ g_n \to \gamma \circ g$ in the sup-norm as $n \to \infty$.

Step 6c. Since $u \circ \gamma \circ f_n = f_n$ and $u \circ \gamma \circ g_n = g$ by the construction of γ, and $f_n \succsim^\circ g_n$ holds by assumption for all $n \in \mathbb{N}$, we have $\gamma \circ f_n \succsim \gamma \circ g_n$ for all $n \in \mathbb{N}$ by the result of Step 4. Since $\succsim$ is continuous, we have $\gamma \circ f \succsim \gamma \circ g$. This implies that $f \succsim^\circ g$, and the claim of Step 6 is proven.

Step 7. The remaining properties of $\succsim^\circ$ follow from the corresponding properties of $\succsim$ easily. □

Lemma 12. *Suppose that $I \subset \mathbb{R}$ is a nondegenerate interval such that* $\min I = 0$*, and suppose that $\succsim^\circ$ is a binary relation on $\mathcal{B}(I)$ that satisfies Reflexivity, Comparative Monotonicity, Scale Invariance, Comparative Improvement, Mirroring, and Continuity. Then, there exists a binary relation $\succsim^\bullet$ on $\mathcal{B}(\mathbb{R}_+)$ that has the same properties and that extends $\succsim^\circ$ in the following sense: For all $f, g \in \mathcal{B}(I)$, $f \succsim^\circ g \Leftrightarrow f \succsim^\bullet g$.*

Proof. Step 1. Let $\succsim^\bullet$ be defined as follows: For all $f, g \in \mathcal{B}(\mathbb{R}_+)$, $f \succsim^\bullet g$ if and only if there exist $\beta \in \mathbb{R}_{++}$ such that $\beta f \succsim^\circ \beta g$.

Step 2. I claim that if $f, g \in \mathcal{B}(\mathbb{R}_+)$ are such that $f \succsim^\bullet g$, then $\gamma f \succsim^\circ \gamma g$ holds for all $\gamma \in \mathbb{R}_{++}$ such that $\gamma f \in \mathcal{B}(I)$ and $\gamma g \in \mathcal{B}(I)$. Indeed, since $f \succsim^\bullet g$, then there exists $\beta > 0$ such that $\beta f \succsim^\circ \beta g$. If $\gamma = \beta$, the claim holds immediately. If $\gamma < \beta$, then $\beta f \succsim^\circ \beta g$ implies $\frac{\gamma}{\beta}\beta f \succsim^\circ \frac{\gamma}{\beta}\beta g$ by Scale Invariance. If $\gamma > \beta$, then $\beta f \succsim^\circ \beta g$ means that $\frac{\beta}{\gamma}\gamma f \succsim^\circ \frac{\beta}{\gamma}\gamma g$, which implies that $\gamma f \succsim^\circ \gamma g$, again by Scale Invariance.

Step 3. Now, I prove Continuity. Suppose that $f, g \in \mathcal{B}(\mathbb{R}_+)$ and sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ in $\mathcal{B}(\mathbb{R}_+)$ are such that $f_n \succsim^{\bullet} g_n$ for all $n \in \mathbb{N}$, and $f_n \to f$ and $g_n \to g$ as $n \to \infty$. Let $\bar{a} := \max\{\max_{\omega \in \Omega} f(\omega), \max_{\omega \in \Omega} g(\omega)\}$, let $\varepsilon \in int\, I$ be arbitrary, and choose $\beta > 0$ such that $\beta \bar{a} + \varepsilon \in I$. Then, observe that $\beta f \in \mathcal{B}(I)$, $\beta g \in \mathcal{B}(I)$, and there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $\beta f_n \in \mathcal{B}(I)$ and $\beta g_n \in \mathcal{B}(I)$. Since $f_n \succsim^{\bullet} g_n$ for all $n \in \mathbb{N}$, by the result of Step 2, we have that $\beta f_n \succsim^{\circ} \beta g_n$ for all $n > N$. By the Continuity property of $\succsim^{\circ}$, we obtain $\beta f \succsim^{\circ} \beta g$, which implies that $f \succsim g$.

Step 4. Given the result of Step 2, the remaining properties of $\succsim^{\bullet}$ follow easily from the corresponding properties of $\succsim^{\circ}$. □

B.2 Proof of Theorem 1

The key technical steps of the proof are presented as separate lemmas.

Lemma 13. *Suppose that $\succsim$ is a binary relation on $\mathcal{B}(\mathbb{R}_+)$ that satisfies Reflexivity, Comparative Monotonicity, Scale Invariance, Comparative Improvement, Mirroring, and Continuity. Let P be defined as*

$$P := \left\{(p, q) \in ba(\Omega) \times ba(\Omega) : \forall_{f,g \in \mathcal{B}(\mathbb{R}_+): f \succsim g} \int f \,\mathrm{d}p - \int g \,\mathrm{d}q \geq 0\right\}.$$

Then,

$$f \succsim g \quad \Leftrightarrow \quad \forall_{(p,q) \in P} \int f \,\mathrm{d}p - \int g \,\mathrm{d}q \geq 0$$

for all $f, g \in \mathcal{B}(\mathbb{R}_+)$.

Proof. Let $K := \{(f, g) \in \mathcal{B}(\mathbb{R}_+)^2 : f \succsim g\}$. Observe that K is a cone by Scale Invariance, convex by Comparative Improvement, and closed by Continuity. Therefore, K is an intersection of all closed half-spaces that contain it (Rockafellar, 1997, Corollary 11.7.1). The claim now follows from the fact that $ba(\Omega)$ is isomorphic to the dual of $\mathcal{B}(\mathbb{R})$. □

Lemma 14. *Suppose that $\succsim$ is a binary relation on $\mathcal{B}(\mathbb{R}_+)$ that satisfies Reflexivity, Comparative Monotonicity, Scale Invariance, Comparative Improvement, Mirroring, and Continuity. Let K' be defined as*

$$K' := \left\{(p, q, \kappa) \in \Delta(\Omega) \times \Delta(\Omega) \times [0, 1] : \forall_{f,g \in \mathcal{B}(\mathbb{R}_+): f \succsim g} \int f \,\mathrm{d}p \geq \kappa \int g \,\mathrm{d}q\right\}.$$

Then,

$$f \succsim g \quad \Leftrightarrow \quad \forall_{(p,q,\kappa)\in K'} \int f \,\mathrm{d}p \geq \kappa \int g \,\mathrm{d}q \tag{7}$$

for all $f, g \in \mathcal{B}(\mathbb{R}_+)$.

Proof. "$\Rightarrow$" part. Fix arbitrary $f, g \in \mathcal{B}(\mathbb{R}_+)$ such that $f \succsim g$, and let P be defined as in Lemma 13. For any $(p, q, \kappa) \in K'$, we have $(p, \kappa q) \in P$ by the definitions of K' and P. The claim now follows from Lemma 13.

"$\Leftarrow$" part. Suppose that $f, g \in \mathcal{B}(\mathbb{R}_+)$ are such that $f \not\succsim g$. My objective is to construct $(p, q, \kappa) \in K'$ such that $\int f \,\mathrm{d}p < \kappa \int g \,\mathrm{d}q$.

Step 1. Since $f \not\succsim g$, then, by Lemma 13, there exist $(p_1, q_1) \in P$ such that $\int f \,\mathrm{d}p_1 < \int g \,\mathrm{d}q_1$. Observe that $p_1(S) \geq 0$ for all $S \in 2^\Omega$ by the definition of P because $(\mathbb{1}_S, 0) \in K$ by Reflexivity and Comparative Monotonicity.

Step 2. Let $S := \{\omega \in \Omega : q_1(\omega) < 0\}$, and let $q_2 \in ba(\Omega)$ be defined as $q_2(\omega) := q_1(\omega)$ if $\omega \notin S$ and $q_2(\omega) := 0$ if $\omega \in S$. Note that $\int g \,\mathrm{d}q_1 \leq \int g \,\mathrm{d}q_2$, and, therefore, $\int f \,\mathrm{d}p < \int g \,\mathrm{d}q_2$.

Now, I claim that $(p_1, q_2) \in P$. To verify that, fix arbitrary $(f', g') \in K$. Let $g'' \in \mathcal{B}(\mathbb{R}_+)$ be defined as $g''(\omega) := g'(\omega)$ if $\omega \notin S$, and $g''(\omega) := 0$ if $\omega \in S$. Then, $(f', g'') \in K$: Indeed, $\frac{1}{2}f' \succsim \frac{1}{2}g' \equiv \frac{1}{2}g'' + \frac{1}{2}(g' - g'')$ by the choice of (f', g') and Scale Invariance, $\frac{1}{2}f' + \frac{1}{2}(g' - g'') \succsim \frac{1}{2}g'' + \frac{1}{2}(g' - g'')$ by Comparative Monotonicity, and, therefore, $\frac{1}{2}f' \succsim \frac{1}{2}g''$ by Mirroring; in turn, $f' \succsim g''$ by Scale Invariance. Now, since $(p_1, q_1) \in P$, we have $\int f' \,\mathrm{d}p_1 \geq \int g'' \,\mathrm{d}q_1$. Notice that $\int g'' \,\mathrm{d}q_1 = \int_{\Omega \setminus S} g' \,\mathrm{d}q_1 = \int_\Omega g' \,\mathrm{d}q_2$, which implies that $\int f' \,\mathrm{d}p_1 \geq \int g' \,\mathrm{d}q_2$. The claim that $(p_1, q_2) \in P$ is now proven.

Step 3. Observe that q_2 is not identically zero: Otherwise, $\int f \,\mathrm{d}p < \int g \,\mathrm{d}q_2$ could not hold. Also, p_1 is not identically zero: For any $S \in 2^\Omega$, $p_1(S) \geq q_2(S)$ because $(p_1, q_2) \in P$ and $(\mathbb{1}_S, \mathbb{1}_S) \in K$ by Reflexivity. Therefore, we can define $p := \frac{1}{p_1(\Omega)} p_1$, $q := \frac{1}{q_2(\Omega)} q_2$, $\kappa := \frac{q_2(\Omega)}{p_1(\Omega)}$, and note that $\kappa \in (0, 1]$. Now, first, $(p_1, q_2) \in P$ implies that $(p, q, \kappa) \in K'$ by the definitions of P and K'. Second, $\int f \,\mathrm{d}p_1 < \int g \,\mathrm{d}q_1$ implies that $\int f \,\mathrm{d}p < \kappa \int g \,\mathrm{d}q$, and the claim is proven. □

Lemma 15. *Suppose that $\succsim$ is a binary relation on $\mathcal{B}(\mathbb{R}_+)$ that satisfies Reflexivity, Comparative Monotonicity, Scale Invariance, Comparative Im-*

provement, Mirroring, and Continuity, and suppose that $f \in B(\mathbb{R}_+)$ and $g \in B(\mathbb{R}_{++})$ are such that $f \not\succsim g$. Then, there exists $h \in B(\mathbb{R}_{++})$ such that

(i) $h(\omega) > f(\omega)$ for all $\omega \in \Omega$, and

(ii) the supporting hyperplane to $\{f' \in \mathcal{B}(\mathbb{R}_+) : f' \succsim g\}$ at h is unique.

Proof. Step 1. Let $C := \{f' \in \mathcal{B}(\mathbb{R}_+) : f' \succsim g\}$, and note that it is closed and convex by Continuity and Comparative Improvement.

Let $h_1 \in \mathcal{B}(\mathbb{R}_+)$ be defined as $h_1(\omega) = \max\{f(\omega), g(\omega)\} + 1$ for all $\omega \in \Omega$. Note that $h_1 \succsim g$ by Comparative Monotonicity. For all $\alpha \in [0,1]$, let $v_\alpha := \alpha h_1 + (1-\alpha) f$. Let $\alpha^* := \sup\{\alpha \in [0,1] : v_\alpha \in T\}$; since $v_0 \notin T$ and T is closed, we have $\alpha^* > 0$.

Step 2. Let $h_2 := v_{1/2\alpha^*}$ and note that $h_2(\omega) - f(\omega) \geq \frac{1}{2}\alpha^*$ for all $\omega \in \Omega$ and $h_2 \notin C$ (see Figure 4). Next, let $C_2 := C \cap \{f' \in \mathcal{B}(\mathbb{R}_+) : \forall_{\omega\in\Omega}\ f'(\omega) \geq f(\omega)\}$, and note that C_2 is convex and closed and $h_2 \notin C_2$. I also claim that C_2 is full-dimensional — i.e., its affine hull is the entire space $\mathcal{B}(\mathbb{R})$. Indeed, fix an arbitrary $\varphi \in \mathcal{B}(\mathbb{R})$, and let $\psi \in \mathcal{B}(\mathbb{R}_+)$ be defined as $\psi(\omega) := \max\{h_1(\omega), \varphi(\omega)\}$. By Comparative Monotonicity, we have $\psi \in C_2$ and $2\psi - \varphi \in C_2$, and, therefore, $\varphi = 2\psi + (-1)(2\psi - \varphi) \in \mathit{aff}(C_2)$.

Step 3. Now, by Rockafellar (1997, Theorem 18.8), there exists $h \in \partial C_2$ such that

(i) the supporting hyperplane to C_2 at h is unique, and

(ii) that supporting hyperplane separates h_2 and C_2.

Step 4. I claim that that $h(\omega) > f(\omega)$ for all $\omega \in \Omega$ and, therefore, $h \in \partial C$. Indeed, suppose, by contradiction, that $h(\omega_0) = f(\omega_0)$ for some $\omega_0 \in \Omega$. Then, for all $f' \in C_2$, we have $f'(\omega_0) \geq h(\omega_0)$, and, therefore, the hyperplane $\{f' \in \mathcal{B}(\mathbb{R}_+) : f'(\omega_0) = h(\omega_0)\}$ is the unique supporting hyperplane to C_2 at h. However, $h_2(\omega) > h(\omega)$, which contradicts the fact that this hyperplane separates h_2 and C_2.

Step 5. Finally, I prove that the supporting hyperplane to C_2 at h is also a supporting hyperplane to C. Let $r \in ba(\Omega)\backslash\{0\}$ be such that the supporting half-space to C_2 at h is $\{f' \in \mathcal{B}(\mathbb{R}_+) : \int f' \,\mathrm{d}r \geq \int h \,\mathrm{d}r\}$, and suppose, by contradiction, that there exists some $f' \in C$ such $\int f' \,\mathrm{d}r < \int h \,\mathrm{d}r$. Then, let $\varepsilon > 0$ be chosen sufficiently small so that $f'' := (1-\varepsilon)h + \varepsilon f'$ satisfies $f''(\omega) > f(\omega)$ for all $\omega \in \Omega$. By Reflexivity and Comparative Improvement, we have

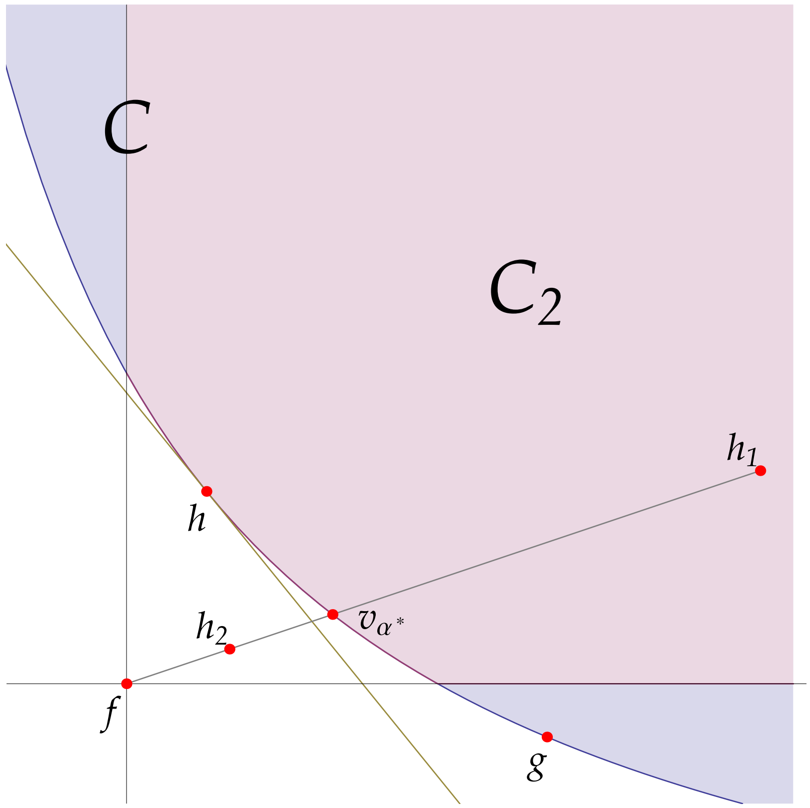

Figure 4.— Illustration of the proof of Lemma 15

$f'' \in C$; since it also satisfies the additional constraints imposed in C_2, we have $f'' \in C_2$. At the same time, $\int f'' \,\mathrm{d}r = (1-\varepsilon)\int h \,\mathrm{d}r + \varepsilon \int f' \,\mathrm{d}r < \int h \,\mathrm{d}r$, which contradicts the definition of the supporting half-space to C_2. $\square$

Lemma 16. *Suppose that $\succsim$ is a binary relation on $\mathcal{B}(\mathbb{R}_+)$ that satisfies Reflexivity, Comparative Monotonicity, Scale Invariance, Comparative Improvement, Mirroring, and Continuity. Let K'' be defined as*

$$K'' := \left\{(p,\kappa) \in \Delta(\Omega) \times [0,1] : \forall_{f,g \in \mathcal{B}(\mathbb{R}_+): f \succsim g} \int f \,\mathrm{d}p \geq \kappa \int g \,\mathrm{d}p \right\}.$$

Then,

$$f \succsim g \quad \Leftrightarrow \quad \forall_{(p,\kappa)\in K''} \int f \,\mathrm{d}p \geq \kappa \int g \,\mathrm{d}p \tag{8}$$

for all $f, g \in \mathcal{B}(\mathbb{R}_+)$.

Proof. "$\Rightarrow$" part. Fix arbitrary $f, g \in \mathcal{B}(\mathbb{R}_+)$ such that $f \gtrsim g$, and let K' be defined as in Lemma 14. For any $(p, \kappa) \in K''$, we have $(p, p, \kappa) \in K'$. The claim now follows from Lemma 14.

"$\Leftarrow$" part. Suppose that $f, g \in \mathcal{B}(\mathbb{R}_+)$ are such that $f \not\gtrsim g$. My objective is to prove that there exist $(p, \kappa) \in K''$ such that $\int f \, \mathrm{d}p < \kappa \int g \, \mathrm{d}p$.

Step 1. I start proving the claim under an additional assumption that $g(\omega) > 0$ for all $\omega \in \Omega$, and I will consider the general case in Step 6.

Step 2. Let $C := \{f' \in \mathcal{B}(\mathbb{R}_+) : f' \gtrsim g\}$, and let $h \in \partial C$ be as delivered by Lemma 15. I claim that there exists $(p, q, \kappa) \in K'$ such that $\int h \, \mathrm{d}p = \kappa \int g \, \mathrm{d}q$. Indeed, let the function $D : K' \to \mathbb{R}$ be defined as $D(p, q, \kappa) := \int h \, \mathrm{d}p - \kappa \int g \, \mathrm{d}q$, and note that $D(p, q, \kappa) \geq 0$ for all $(p, q, \kappa) \in K'$ by Lemma 14. Note that $K' \subseteq \Delta(\Omega) \times \Delta(\Omega) \times [0, 1]$ and is closed as an intersection of closed sets; therefore, it is compact. Since D is continuous, it achieves its minimum on K'. Let $\beta := \min D(K')$, and suppose that $\beta > 0$. Then, for all $h' \in \mathcal{B}(\mathbb{R}_+)$ such that $\|h' - h\| < \beta$, we have $\int h' \, \mathrm{d}p - \kappa \int g \, \mathrm{d}q \geq 0$ for all $(p, q, \kappa) \in K'$, and, therefore, by Lemma 14, $h' \gtrsim g$, which contradicts the fact that $h \in \partial C$. We conclude that $\beta = 0$. Now, pick any $(p, q, \kappa) \in K'$ such that $D(p, q, \kappa) = 0$, and observe that it satisfies the claimed equality.

Step 3. Let $L_p : \mathcal{B}(\mathbb{R}_+) \to \mathbb{R}$ be a linear functional defined as $L_p(v) := \int v \, \mathrm{d}p$. Observe that the hyperplane $L_p(\cdot) = L_p(h)$ is a supporting hyperplane to C at h: Indeed, $(p, q, \kappa) \in K'$ implies that, for any $f' \in C$, we have $\int f' \, \mathrm{d}p \geq \kappa \int g \, \mathrm{d}q$; therefore, since $\kappa \int g \, \mathrm{d}q = \int h \, \mathrm{d}p$ by the choice of h, we obtain $L_p(f') \geq L_p(h)$.

Step 4. Now, I claim that the hyperplane $L_q(\cdot) = L_q(h)$, where $L_q(v) := \int v \, \mathrm{d}q$ for all $v \in \mathcal{B}(\mathbb{R}_+)$, is also a supporting hyperplane to C at h.

To prove that, suppose, by contradiction, that there exists $f_1 \in C$ such that $L_q(f_1) < L_q(h)$.

Let $M := \|f_1 - h\|$, $m := \min_{\omega \in \Omega} g(\omega)$, and note that $m > 0$ by the assumption made in Step 1. Let α be chosen arbitrarily in $(0, \frac{m}{M})$, and let f_2 be defined as $f_2 := \alpha f_1 + (1 - \alpha) h$. Since $f_1 \gtrsim g$ and $h \gtrsim g$, we have $f_2 \gtrsim g$ by Comparative Improvement. Next, let $v \in \mathcal{B}(\mathbb{R}_+)$ be defined as $v(\omega) := \min\{h(\omega), f_2(\omega)\}$ for all $\omega \in \Omega$, and $w := g - (f_2 - v)$. Note that, for all $\omega \in \Omega$, $(f_2 - v)(\omega) \leq \alpha M$ and, by the choice of α, $w(\omega) \geq 0$. Now, observe that

$$h = \frac{1}{2}v + \frac{1}{2}(v + 2(h - v)), \qquad f_2 = \frac{1}{2}v + \frac{1}{2}(v + 2(f_2 - v)),$$
$$g = \frac{1}{2}w + \frac{1}{2}(w + 2(f_2 - v)).$$

Therefore, by Mirroring, we have $h \gtrsim \frac{1}{2}w + \frac{1}{2}(w + 2(h - v))$, which, after rearrangement, implies $h \gtrsim g - f_2 + h$.

Since $(p, q, \kappa) \in K'$, we have $\int h \, \mathrm{d}p \geq \kappa \int (g - f_2 + h) \, \mathrm{d}q$. Given that $\int h \, \mathrm{d}p = \kappa \int g \, \mathrm{d}q$, we obtain $0 \geq \int (h - f_2) \, \mathrm{d}q$, and, in turn, $0 \geq \int (h - f_1) \, \mathrm{d}q$, which contradicts the choice of f_1.

Step 5. Since the supporting hyperplane to C at h is unique, we conclude that $p = q$, and that $(p, p, \kappa) \in K'$. Since $h(\omega) > f(\omega)$ for all $\omega \in \Omega$ by the construction of h, we have $\int f \, \mathrm{d}p < \int h \, \mathrm{d}p = \kappa \int g \, \mathrm{d}p$. This completes the proof of the claim if $g(\omega) > 0$ for all $\omega \in \Omega$.

Step 6. Finally, I prove the implication $f \not\gtrsim g \Rightarrow \exists_{(p,\kappa) \in K''} \int f \, \mathrm{d}p < \kappa \int g \, \mathrm{d}p$ in the general case. Fix arbitrary $f, g \in \mathcal{B}(\mathbb{R}_+)$ such that $f \not\gtrsim g$. By Continuity, there exists $\varepsilon > 0$ sufficiently small such that $f + \varepsilon \mathbb{1}_\Omega \not\gtrsim g + \varepsilon \mathbb{1}_\Omega$. By the result of previous steps, there exist $(p, \kappa) \in K''$ such that $\int (f + \mathbb{1}_\Omega) \, \mathrm{d}p < \kappa \int (g + \mathbb{1}_\Omega) \, \mathrm{d}p$; since $\kappa \leq 1$, this implies that $\int f \, \mathrm{d}p < \kappa \int g \, \mathrm{d}p$. □

Proof of Theorem 1. *Only if* part. Suppose that $c : \mathcal{C} \rightrightarrows \mathcal{F}$ is a correspondence that satisfies axioms (A1)–(A10).

Step 1. By Proposition 10, there exist two binary relations $\gtrsim^*$ and $\gtrsim$ on $\mathcal{F}$ such that $c(S, d) = \mathrm{Max}_{\gtrsim^*}\{f \mid f \in S, f \gtrsim d\}$, and such that

(i) $\gtrsim^*$ satisfies Reflexivity, Transitivity, Completeness, Monotonicity, Independence, Continuity, and Nondegeneracy;

(ii) $\gtrsim$ satisfies Reflexivity, Worst Element, Scale Invariance, Comparative Improvement, Mirroring, and Continuity;

(iii) $\gtrsim^*$ and $\gtrsim$ jointly have the following properties:

   (a) For all $f, g \in \mathcal{F}$, $f \gtrsim g$ implies $f \gtrsim^* g$.

   (b) For all $d \in \mathcal{F}$ and all $f, g \in \mathcal{F}$ such that $g(\omega) \gtrsim^* f(\omega)$ for all $\omega \in \Omega$, we have $f \gtrsim d \Rightarrow g \gtrsim d$.

By the Anscombe-Aumann expected utility theorem, there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a prior $p \in \Delta(\Omega)$ such that, for all $f, g \in \mathcal{F}$, $f \succsim^* g \Leftrightarrow \int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p$. Moreover, u must be continuous because $\succsim^*$ is continuous and u has a convex range. Since u is defined up to a positive affine transformation, it can be assumed without loss of generality that $\min u(X) = u(x_*) = 0$.

Step 2. Now, u and $\succsim$ satisfy the conditions of Lemma 11. Let $\succsim^\circ$ be a binary relation on $\mathcal{B}(u(X))$ such that $f \succsim d \Leftrightarrow (u \circ f) \succsim^\circ (u \circ d)$ for all $f, d \in \mathcal{F}$, as defined by that lemma. By Lemma 12, the relation $\succsim^\circ$ can be extended to a relation $\succsim^\bullet$ on $\mathcal{B}(\mathbb{R}_+)$.

Step 3. By Lemma 16, there exists a nonempty set $K'' \subseteq \Delta(\Omega) \times [0,1]$ such that

$$f \succsim d \quad \Leftrightarrow \quad (u \circ f) \succsim^\bullet (u \circ d) \quad \Leftrightarrow \quad \forall_{(q,\kappa) \in K''} \int (u \circ f)\,\mathrm{d}q \geq \kappa \int (u \circ d)\,\mathrm{d}q$$

for all $f, d \in \mathcal{F}$.

Step 4. Let $k(q) := \sup\{\kappa \in [0,1] : (q,\kappa) \in K''\}$ for all $q \in \Delta(\Omega)$. Since K'' is an intersection of closed sets and, therefore, closed, we have $(q, k(q)) \in K''$ for all $q \in \Delta(\Omega)$. Moreover, for all $q \in \Delta(\Omega)$ and $\kappa \in [0, k(q)]$, we have $(q, \kappa) \in K''$. Therefore, representation (1) holds.

Only if part. Suppose that c is a correspondence defined by representation (1), where u, p, and k satisfy the conditions stated in the theorem. My objective is to prove that c satisfies all the listed axioms.

Choice Correspondence: It is immediate that $c(S,d) \subseteq S$ for all $(S,d) \in \mathcal{C}$. It is also nonempty because d always satisfies all the constraints.

WARP: Suppose that $(S,d), (T,d) \in \mathcal{C}$, and $f, g \in \mathcal{F}$ are such that $f, g \in S \cap T$, $f \in c(S,d)$, and $g \in c(T,d)$. Since f and g are chosen given the default d, they must satisfy all the constraints. We also have that $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p$ and $\int (u \circ g)\,\mathrm{d}p \geq \int (u \circ h)\,\mathrm{d}p$ for all $h \in T$ such that $\int (u \circ h)\,\mathrm{d}q \geq k(q) \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$. Then, it is immediate that $f \in c(T,d)$.

Monotonicity: Suppose that $(S,d) \in \mathcal{C}$ and $f, g \in \mathcal{F}$ are such that $f \in c(S \cup \{f\}, d)$ and $g(\omega) \in c(\{g(\omega), f(\omega), x_*\}, x_*)$ for all $\omega \in \Omega$. The latter implies that $u(g(\omega)) \geq u(f(\omega))$ for all $\omega \in \Omega$, and, therefore, $\int (u \circ g)\,\mathrm{d}p \geq$

$\int (u \circ f)\, \mathrm{d}p$, which, in turn, implies that $\int (u \circ g)\, \mathrm{d}p \geq \int (u \circ h)\, \mathrm{d}p$ for all $h \in T$ such that $\int (u \circ h)\, \mathrm{d}q \geq k(q) \int (u \circ d)\, \mathrm{d}q$ for all $q \in \Delta(\Omega)$. Since we also have $\int (u \circ g)\, \mathrm{d}q \geq \int (u \circ f)\, \mathrm{d}q \geq k(q) \int (u \circ d)\, \mathrm{d}q$ for all $q \in \Delta(\Omega)$, it follows that $g \in c(S \cup \{g\}, d)$.

Qualified Independence: Suppose that $f, g, h, d \in \mathcal{F}$ are such that $f \in c(\{f, d\}, d)$, $g \in c(\{g, d\}, d)$, and $h \in c(\{h, d\}, d)$, and suppose that $\alpha \in (0, 1]$. Then, f and g must satisfy all the maximization constraints in the representation, as well as the mixtures $\alpha f + (1-\alpha)h$ and $\alpha g + (1-\alpha)h$. The claim of the axiom, then, follows from the linearity of the integral.

Scale Invariance: This axiom follows immediately since $u(x_*) = 0$, u is affine, and the constraints and the objective function of the optimization are linear.

Comparative Improvement: Suppose that $f, g, h', h \in \mathcal{F}$ are such that $f \in c(\{f, g\}, g)$ and $h' \in c(\{h', h\}, h)$, and fix an arbitrary $\alpha \in [0, 1]$. Then, it must be that

$$\begin{aligned}
&\int (u \circ f)\, \mathrm{d}p \geq \int (u \circ g)\, \mathrm{d}p, \quad \int (u \circ f)\, \mathrm{d}q \geq k(q) \int (u \circ g)\, \mathrm{d}q,\\
&\int (u \circ h')\, \mathrm{d}p \geq \int (u \circ h)\, \mathrm{d}p, \quad \int (u \circ h')\, \mathrm{d}q \geq k(q) \int (u \circ h)\, \mathrm{d}q\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } q \in \Delta(\Omega).
\end{aligned}$$

Then, taking the convex combination of the first and the second rows with the weights of α and $1-\alpha$, respectively, we obtain the claimed: $\alpha f + (1-\alpha)h' \in c(\{\alpha f + (1-\alpha)h', \alpha g + (1-\alpha)h\}, \alpha g + (1-\alpha)h)$.

Default Option Mirroring Suppose that $f, d, h', h \in \mathcal{F}$ and $\alpha \in [0, 1]$ are such that

$$\begin{aligned}
\alpha f + (1-\alpha)h &\in c(\{\alpha f + (1-\alpha)h, \alpha d + (1-\alpha)h')\}, \alpha d + (1-\alpha)h'),\\
\alpha f + (1-\alpha)h' &\in c(\{\alpha f + (1-\alpha)h', \alpha d + (1-\alpha)h')\}, \alpha d + (1-\alpha)h').
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\alpha \int (u \circ f)\, \mathrm{d}q + (1-\alpha) \int (u \circ h)\, \mathrm{d}q \geq\\
&\qquad\qquad k(q)\, \alpha \int (u \circ d)\, \mathrm{d}q + k(q)(1-\alpha) \int (u \circ h')\, \mathrm{d}q,\\
&\alpha \int (u \circ f)\, \mathrm{d}q + (1-\alpha) \int (u \circ h')\, \mathrm{d}q \geq\\
&\qquad\qquad k(q)\, \alpha \int (u \circ d)\, \mathrm{d}q + k(q)(1-\alpha) \int (u \circ h')\, \mathrm{d}q
\end{aligned}$$

for all $q \in \Delta(\Omega)$. Taking a convex combination of these two inequalities with the weights of $1 - k(q)$ and $k(q)$, respectively, leads to

$$\alpha \int (u \circ f)\,\mathrm{d}q + (1 - k(q))(1 - \alpha) \int (u \circ h)\,\mathrm{d}q \geq k(q)\,\alpha \int (u \circ d)\,\mathrm{d}q$$

for all $q \in \Delta(\Omega)$. Since this inequality holds also, in particular, for $q = p$ with $k(p) = 1$, we obtain after rearrangement $\alpha f + (1 - \alpha)h \in c(\{\alpha f + (1 - \alpha)h, \alpha d + (1 - \alpha)h\}, \alpha d + (1 - \alpha)h)$.

Continuity: Suppose that $f, g, d \in \mathcal{F}$ and sequences $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$, $(d_n)_{n=1}^{\infty}$ in $\mathcal{F}$ are such that $f_n \to f$, $g_n \to g$, $d_n \to d$ as $n \to \infty$, and $f_n \in c(\{f_n, g_n, d_n\}, d_n)$ and $g_n \in c(\{g_n, d_n\})$ for all $n \in \mathbb{N}$. then $f \in c(\{f, g, d\}, d)$. This means that, for all $n \in \mathbb{N}$, we have $\int (u \circ f_n)\,\mathrm{d}p \geq \int (u \circ g_n)\,\mathrm{d}p$, $\int (u \circ f_n)\,\mathrm{d}p \geq \int (u \circ d_n)\,\mathrm{d}p$, and $\int (u \circ f_n)\,\mathrm{d}q \geq \int (u \circ d_n)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$. Then, by the continuity of u and the integral, we get $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p$, $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ d)\,\mathrm{d}p$, and $\int (u \circ f)\,\mathrm{d}q \geq \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$, which proves the axiom.

Worst Element and *Nondegeneracy* axioms follow immediately from the properties of u. □

B.3 Proofs of the remaining results

Lemma 17. *Suppose that a choice correspondence $c : \mathcal{C} \rightrightarrows \mathcal{F}$ satisfies axioms (A1)–(A10), and (u, p, k) is its representation. Suppose, also, that $q_0 \in \Delta(\Omega)$ and $\kappa \in [0, 1]$ are such that*

$$f \in c(\{f, d\}, d) \quad \Rightarrow \quad \int (u \circ f)\,\mathrm{d}q_0 \geq \kappa \int (u \circ d)\,\mathrm{d}q_0$$

for all $f \in \mathcal{F}$, and let $k' : \Delta(\Omega) \to [0, 1]$ be defined as $k'(q) = k(q)$ for all $q \neq q_0$, and $k'(q_0) = \max\{k(q_0), \kappa\}$. Then, (u, p, k') is also a representation of c.

Proof. Fix an arbitrary decision problem $(S, d) \in \mathcal{C}$, and suppose, first, that $f \in c(S, d)$. My objective is to prove that

$$\begin{aligned} & f \in \operatorname*{Arg\,max}_{h \in S} \int (u \circ h)\,\mathrm{d}p \\ & \quad \text{s.t. } \int (u \circ h)\,\mathrm{d}q \geq k'(q) \int (u \circ d)\,\mathrm{d}q \text{ for all } q \in \Delta(\Omega). \end{aligned}$$

Indeed, observe that $\int (u \circ f)\,\mathrm{d}q \geq k(q) \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$ because (u, p, k) is a representation of c. Moreover, by WARP, $f \in c(S, d)$ implies $f \in c(\{f, d\}, d)$, and, therefore, $\int (u \circ f)\,\mathrm{d}q_0 \geq \kappa \int (u \circ d)\,\mathrm{d}q_0$. Together, this implies that $\int (u \circ f)\,\mathrm{d}q \geq k'(q) \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$. Finally, since (u, p, k) is a representation of c, we have $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p$ for all $g \in \mathcal{F}_{d,k}$, where $\mathcal{F}_{d,\zeta}$ is defined for each $\zeta \in [0,1]^{\Delta(\Omega)}$ as

$$\mathcal{F}_{d,\zeta} := \left\{ g \in \mathcal{F} : \int (u \circ g)\,\mathrm{d}q \geq \zeta(q) \int (u \circ d)\,\mathrm{d}q \text{ for all } q \in \Delta(\Omega) \right\}.$$

Then, per force, $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p$ for all $g \in \mathcal{F}_{d,k'}$.

Conversely, suppose that $f \in S$ is such that

$$\begin{aligned} f \in \operatorname*{Arg\,max}_{h \in S} & \int (u \circ h)\,\mathrm{d}p \\ \text{s.t. } & \int (u \circ h)\,\mathrm{d}q \geq k'(q) \int (u \circ d)\,\mathrm{d}q \text{ for all } q \in \Delta(\Omega), \end{aligned}$$

and, by contradiction, suppose that $f \notin c(S, d)$. Let $h \in c(S, d)$ be arbitrary. Now, observe that $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p$ for all $g \in \mathcal{F}_{d,k}$, and, therefore, it must be that $\int (u \circ h)\,\mathrm{d}p > \int (u \circ f)\,\mathrm{d}p$ and $h \in \mathcal{F}_{d,k}$. Since $h \in c(S, d)$, we have, by WARP, $h \in c(\{h, d\}, d)$. Therefore, $\int (u \circ h)\,\mathrm{d}q_0 \geq \kappa \int (u \circ d)\,\mathrm{d}q_0$, which implies that $h \in \mathcal{F}_{d,k'}$. Together with the earlier inequality, this contradicts the choice of f as a constrained maximizer. $\square$

Proof of Theorem 2. Observe, first, that $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ g)\,\mathrm{d}p \Leftrightarrow f \in c(\{f, g, x_*\}, x_*) \Leftrightarrow \int (u' \circ f)\,\mathrm{d}p' \geq \int (u' \circ g)\,\mathrm{d}p'$ for all $f, g \in \mathcal{F}$. Therefore, by the uniqueness of the expected utility representation, we have $p' = p$ and $u' = \beta u + \alpha$ for some $\beta > 0$ and $\alpha \in \mathbb{R}$. Given the normalization $u(x_*) = 0 = u'(x_*)$, we conclude that $\alpha = 0$. The fact that $(u, p, \max(k, k'))$ is a representation of c follows from Lemma 17.

Next, let $\mathcal{R}$ denote the set of all tuples (u', p', k') that represent c, and let $k^* : \Delta(\Omega) \to [0,1]$ be defined as $k^*(q) = \sup_{(u',p',k') \in \mathcal{R}} k'(q)$ for all $q \in \Delta(\Omega)$. As follows from the result of the previous paragraph, for any $f, d \in \mathcal{F}$ such that $f \in c(\{f, d\}, d)$, we have $\int (u \circ f)\,\mathrm{d}q \geq k'(q) \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$ and for all $(u', p', k') \in \mathcal{R}$. Therefore, for any $f, d \in \mathcal{F}$ such that $f \in c(\{f, d\}, d)$, we have $\int (u \circ f)\,\mathrm{d}q \geq k^*(q) \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$. Then, by Lemma 17, (u, p, k^*) is a representation of c. $\square$

Proof of Observation 3. The "if" part of the claim is trivial. To prove the "only if" part, suppose that a choice correspondence c admits a representation

$$c(S,d) = \operatorname*{Arg\,max}_{f \in S} \int u(f(\omega))\,\mathrm{d}p(\omega) \tag{9}$$

for some nonconstant, continuous, and affine $u : X \to \mathbb{R}$ such that $\min u(X) = 0$, and $p \in \delta(\Omega)$, and (u', p', k') is another guaranteed expected utility representation of c. As follows from Theorem 2, we have $u' = \beta u$ for some $\beta > 0$, and $p' = p$.

Suppose, by contradiction, that $k'(q) > 0$ for some $q \in \Delta(\Omega)\backslash\{p\}$. Let $E \subset \Omega$ be such that $q(E) \neq p(E)$, and assume without loss of generality that $q(E) < p(E)$. Next, let $x_*, x^* \in X$ be chosen such that $u(x_*) = 0$ and $u(x^*) > 0$, which is possible because $\min u(X) = 0$ and u is nonconstant, and let $\tilde{x} \in X$ be chosen such that $u(\tilde{x}) = p(E)u(x^*)$, which is possible because u is affine. Now, consider act $f \in \mathcal{F}$ defined as $f(\omega) = x^*$ if $\omega \in E$ and $f(\omega) = x_*$ if $\omega \notin E$. As follows from (9), we have $f \in c(\{f, \tilde{x}\}, \tilde{x})$. At the same time, $\int (u' \circ f)\,\mathrm{d}q - k'(q)u'(\tilde{x}) = q(E)\beta u(x^*) - k'(q)p(E)\beta u(x^*) < 0$, which contradicts the fact that (u', p', k') is a representation of c and $f \in c(\{f, \tilde{x}\}, \tilde{x})$. $\square$

Proof of Proposition 4. *If* part. Suppose that $u_2 = \beta u_1$ for some $\beta > 0$ and $k_2^*(q) \geq k_1^*(q)$ for all $q \in \Delta(\Omega)$, and fix an arbitrary choice problem $(S, d) \in \mathcal{C}$ such that $\{d\} = c_1(S, d)$. Now, suppose, by contradiction, that there exists $f \in S$, $f \neq d$, such that $f \in c_2(S, d)$. Since $\{d\} = c_1(S, d)$, we have two possibilities: $\int (u_1 \circ f)\,\mathrm{d}q < k_1^*(q) \int (u_1 \circ d)\,\mathrm{d}q$ for some $q \in \Delta(\Omega)$ or $\int (u_1 \circ d)\,\mathrm{d}p > \int (u_1 \circ f)\,\mathrm{d}p$. In the first case, that inequality implies $\int (u_1 \circ f)\,\mathrm{d}q < k_2^*(q) \int (u_1 \circ d)\,\mathrm{d}q$; in the second case, note that $k_1^*(p) = 1$, and, hence, $k_2^*(p) = 1$ and $\int (u_1 \circ f)\,\mathrm{d}p < k_2^*(p) \int (u_1 \circ d)\,\mathrm{d}p$. Both these cases lead to a contradiction with $f \in c_2(S, d)$.

Only if part. Suppose that c_2 is more mistake averse than c_1. First, observe that, for any $x, y \in X$, $u_2(x) \geq u_2(y) \Rightarrow x \in c_2(\{x, y\}, y) \Rightarrow x \in c_1(\{x, y\}, y) \Rightarrow u_1(x) \geq u_1(y)$. Then, it must be that $u_2 = \beta u_1 + \alpha$ for some $\beta > 0$ and $\alpha \in \mathbb{R}$,and, given the normalization $\min u_1(X) = \min u_2(X) = 0$, it follows that $\alpha = 0$.

Next, fix an arbitrary $q \in \Delta(\Omega)$. For any $(f,d) \in \mathcal{F}^2$ such that $f \in c_2(\{f,d\},d)$, we have $f \in c_1(\{f,d\},d)$, and, therefore, $\int (u_1 \circ f)\,\mathrm{d}q \geq k_1^*(q) \int (u_1 \circ f)\,\mathrm{d}q$. By Lemma 17, this implies that $k_2^*(q) \geq k_1^*(q)$. □

Proof of Proposition 5. Suppose that $k_2(q) \geq k_1(q)$ for all $q \in \Delta(\Omega)$, and fix an arbitrary choice problem $(S,d) \in \mathcal{C}$ such that $\{d\} = c_1(S,d)$. Now, suppose, by contradiction, that $\{d\} \neq c_2(S,d)$, and, therefore, there exists $f \in S$ such that $f \neq d$ and $f \in c_2(S,d)$. Given the representation of c_2, we have $\int (u \circ f)\,\mathrm{d}q \geq k_2(q) \int (u \circ d)\,\mathrm{d}q$ for all $q \in \Delta(\Omega)$, and $\int (u \circ f)\,\mathrm{d}p \geq \int (u \circ d)\,\mathrm{d}p$. In turn, given the representation of c_1, we obtain that $f \in c_1(S,d)$, a contradiction. □

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