Decision-Making under Subjective Risk: Toward a General Theory of Pessimism*

Anna Gumen† Efe A. Ok‡ Andrei Savochkin§

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Abstract

The primary objective of this paper is to develop a framework in which a decision-maker may have subjective beliefs about the "riskiness" of prospects, even though the risk structure of these prospects is objectively specified. Put differently, we investigate preferences over risky alternatives by postulating that such preferences arise from more basic preferences that act on the subjective transformations of these prospects. This allows *deriving* a theory of preferences over lotteries with distorted probabilities and provides information about the structure of such distortions. In particular, we are able to formulate a behavioral trait such as "pessimism" in the context of risk (independently of any sort of utility representation) as a particular manifestation of the uncertainty aversion phenomenon. Our framework also provides a strong connection between the notions of aversion to ambiguity and risk which are regarded as distinct traits in decision theory.

JEL Classification: D11, D81.
Keywords: Distortion of Probabilities, Pessimism/Optimism, Non-Expected Utility Theory, Uncertainty Aversion, Risk Aversion.

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†Department of Economics, New York University. E-mail: anna.gumen@nyu.edu.

‡Department of Economics, New York University, 19 West 4th Street, New York, NY 10012. E-mail: efe.ok@nyu.edu.

§Collegio Carlo Alberto, Via Real Collegio 30, 10024 Moncalieri (TO), Italy. E-mail: andrei.savochkin@carloalberto.org.

Suam habet fortuna rationem.[1]
Petronius

1 Introduction

Motivation. We hear the following types of statements routinely in daily discourse:

"Roll the dice, I feel lucky today."

"I'm really not lucky when it comes to slot machines."

"There was 10% chance I could lose, but I knew I would."

There are even commonly used narratives such as "Murphy's law," or "50-50-90 rule: If the chance of something good happens is 50-50, then something bad happens 90% of the time."

It is, of course, important to understand the behavior of people who say such things and make decisions accordingly, at least from a descriptive viewpoint. Yet, to a rational decision theorist, it is not an easy matter to make sense of such sentiments, and even to take them seriously. After all, a toss of a fair coin that pays \$10 if heads come up and -\$5 otherwise, is supposed to be just that; a probability distribution $\mathbf{p}$ with $\mathbf{p}(-5) = \frac{1}{2} = \mathbf{p}(10)$, whoever plays it. It is not that decision theorists are strangers to traits like pessimism or optimism, but those notions are readily meaningful only when there is room for subjective assessment of likelihoods of outcomes, not when these likelihoods are provided in an objective sense. In the example of a toss of a fair coin, probabilities are presumed to be known – the situation is one of risk, not of uncertainty, to use the distinction made famously by Frank Knight.

One sensible way out of this conundrum is to recognize that an individual holds the right to *subjectively* evaluate the likelihood of outcomes in a lottery even though she understands that these likelihoods are indeed *objectively* given. This evaluation will, of course, take the objectively given probabilities into account, but it does not have to yield "beliefs" that match these probabilities exactly. To wit, in the fair coin tossing example above, we propose to think of an individual as one who is facing an act f on the state space $\{H, T\}$ with $f(H) = 10$ and $f(T) = -5$. What distinguishes this from a purely Savagean outlook is that the individual has the information that the (objective) probabilities of H and T are $\frac{1}{2}$. Put differently, while the von Neumann-Morgenstern formulation of this bet is μ, the uniform probability distribution on $\{H, T\}$, and the purely Savagean formulation of it is simply f (that would model a situation in which nothing is known about the likelihood of outcomes), we propose that the individual looks at this bet in a hybrid way as (μ, f).[2]

In a nutshell, the main goal of the present paper is to investigate preferences over (objective) risky prospects by postulating that such preferences arise from more basic preferences that act on the subjective transformations of these prospects. As we shall see, this approach allows one to "derive" a theory of preferences over lotteries with distorted probabilities, and provides information about the structure of such distortions.

[1] Chance has its reasons.

[2] To illustrate, in Figure 1, the left-most part represents the von Neumann-Morgenstern formulation of this lottery, the center one the purely Savagean formulation, and the right-most part corresponds to the hybrid formulation we propose here.

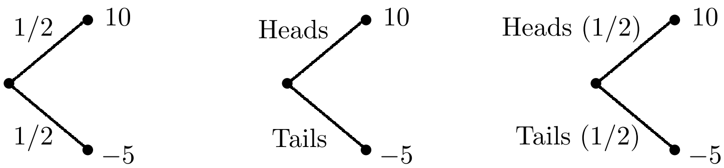

Figure 1

The idea of "distorted probabilities" is, of course, not new. It dates back at least to the works of Preston and Baratta (1948) and Edwards (1954) in psychology, and Kahneman and Tversky (1979) in economics. Indeed, non-expected utility theory provides a number of models in which the decision maker evaluates a lottery by means of an expected utility computation, one in which probabilities of outcomes are not necessarily the ones given in the lottery. There is, however, a major difference between our approach and non-expected utility theory. The latter theory does not have room for looking at these probability distortions at the level of primitives. Instead, it derives them by means of structural postulates that weaken the classical independence axiom (so that the theory conforms with the Allais paradox, common ratio effect, etc.), thereby providing behavioral tests for checking whether or not these distortions take place in some consistent manner. While it is obviously useful, this approach does not attribute any behavioral meaning to such distortions, they are meaningful only within specific models. As we shall see, this makes it quite difficult to talk about traits such as pessimism or optimism at a general level. By contrast, what we wish to do in this paper is to allow an agent to view his "luck" as different than the objectively given probabilities at the level of primitives (not only as an interpretation of the representation of the preferences). In other words, we introduce the possibility of subjective evaluation of the likelihood of outcomes in a lottery at the modeling stage, thereby giving a Savagean outlook to risk preferences; hence the term *subjective risk preferences.* This approach permits the analyst to look at distorted probabilities of non-expected utility theory as the "beliefs" of the decision maker about the various outcomes of a given lottery. As such, it builds a bridge through which the insights obtained in the theory of decision-making under uncertainty can be carried into the realm of choice under risk.

An Outline of the Subjective Risk Model. In Section 2, we introduce the main ingredients of our model. First, we introduce a natural state space Ω that allows one to view any given monetary lottery as an act. (A state describes what occurs in every lottery in the world, providing a complete resolution of uncertainty.) The objectively given probabilities of a lottery induce a probability distribution on a (finite) partition of Ω. Consequently, every lottery $\mathbf{p}$ is transformed into what we call a *degenerate info-act*, (μ, f), where f takes value x in all states where x is the outcome of $\mathbf{p}$, and μ is a probability distribution on Ω that tells us the probability of the event that "x is the outcome of $\mathbf{p}$" is $\mathbf{p}(x)$. Second, we postulate that the decision maker has preferences over all degenerate info-acts, and she prefers one lottery over another iff she prefers the info-act that corresponds to the former lottery to the info-act that corresponds to the latter.

This is, of course, nothing but relabelling so far. Next, to get some mileage from the approach, we extend the notion of a degenerate info-act into what we simply call an *info-act* (μ, f), by allowing f to pay out consequences other than the payoffs of the involved lottery when a state is realized. (Such an info-act pays a particular consequence (which need not be x) in the event that "x is the outcome of $\mathbf{p}$".) To make use of the Anscombe-Aumann theory, we take such consequences themselves as lotteries, and hypothesize that the agent has well-defined preferences over these info-acts as well. This way, the space of lotteries embeds in the space of info-acts (but the latter is much larger than the former), and we can read off one's risk preferences from her preferences over info-acts (by looking at how she ranks degenerate info-acts), but not conversely. In fact, under an unexceptionably weak monotonicity hypothesis, we show that the latter preferences yield risk preferences that are represented by expected utility, albeit, with distorted probabilities (Proposition 1). It is in this sense that the subjective risk theory "derives" the distortions of likelihood of outcomes in a lottery from the preferences of a person under uncertainty.[3]

One way of thinking about preferences over info-acts is as the "model of the mind" of the agent in the context of the evaluation of uncertainty in general. Then, if one is prepared to make some general assumptions on the structure of this "model of the mind," we may gather insights in terms of the risk preferences of the agent. For instance, the first main result of this paper shows that, under a standard monotonicity hypothesis, whether or not a decision maker distorts objective probabilities (and then use the distorted probabilities to make expected utility computations) depends vitally on her attitude toward ambiguity. Put more concretely, ambiguity neutrality of a decision maker ensures that this person does not distort the probabilities and acts as a standard expected utility maximizer (Theorem 2). In a formal sense, therefore, our approach entails that probability distortions of non-expected utility theory can be seen to arise from one's lack of neutrality toward ambiguity, thereby allowing a viewpoint in which the Allais paradox emerges as a "special case" of certain types of the Ellsberg paradox. (Notably, this statement cannot even be formulated in the context of non-expected utility theory.)

Pushing this point further, we can also deduce certain properties of one's risk preferences from her attitudes toward uncertainty. In particular, we prove in Theorem 9 that risk aversion that is caused by probability distortions is a consequence of one's uncertainty aversion at large (in the sense that if one's subjective risk preferences are uncertainty averse, then the corresponding risk preferences must be (probabilistically) risk averse). This observation brings together two empirically meaningful phenomena, uncertainty aversion and risk aversion, which exist in formally disparate realms, through subjective risk theory. It makes it formally meaningful to assert that one's (probabilistic) risk aversion is a consequence of her ambiguity aversion (again a statement that cannot be formulated in the context of non-expected utility theory).

A General Formulation of Pessimism for Risk Preferences. A major advantage of the "subjective

[3] A quick remark about the observability of preferences over info-acts is in order. It is plain that one cannot identify the entirety of the preferences of a person over info-acts from her choices over risky prospects; the latter type of choices identify only the part of these preferences over degenerate acts. Consequently, insofar the analysis is confined *only* to lottery choices, preferences over info-acts are only partially observable. But this does not mean that such preferences are rather artificial. As info-acts are observable objects, one can ask an agent directly to rank info-act pairs, thereby deducing her info-act preferences completely.

risk" approach is to allow the analyst to formulate certain subjective behavioral traits, such as pessimism, in the context of risk. Pessimism is viewed in non-expected utility theory as some form of over(under)weighting of the probability of bad (good) outcomes; it is commonly defined for the general weighted EU model by using the probability weighting function of that model. Unfortunately, due to the non-uniqueness of probability weighting functions that represent a preference relation as in that model, this way of defining pessimism is behaviorally meaningful only in some special cases of the general weighted EU model with strong uniqueness properties. In particular, this definition (given formally in Section 4.3) is behaviorally meaningful, and it delivers an intuitively appealing conceptualization of pessimism in the context of rank-dependent utility (RDU) model. Yet, even when it is meaningful, this approach turns out to be rather coarse. For instance, as we prove in Proposition 8, in the context of disappointment averse preferences of Gul (1991), this way of looking at pessimism fails to classify *any* preference as pessimistic or optimistic, other than the trivial case of expected utility preferences.

It is important to note that such difficulties do not arise in the context of choice under uncertainty. Indeed, in that context, there is already a widely used notion of pessimism in the context of uncertainty; this is the notion of *uncertainty aversion* (that is, preference for hedging). Roughly speaking, what we propose to do here is to classify a risk preference as pessimistic if that preference is the projection of an uncertainty averse preference over the info-acts. As we explain in some detail in Section 4.1, there are also difficulties in this approach related to non-uniqueness, for, in general, one cannot in general identify one's preferences over info-acts from her (observable) risk preferences. However, if one is prepared to make some general assumptions on the structure of the presumed "model of the mind," that is, on one's preferences over info-acts, the difficulty may be circumvented. In particular, if we restrict our attention to a class of preferences over info-acts that are uniquely identified by their restriction to the set of degenerate info-acts – such a class of preferences is called *viable* in this paper – we do achieve a one-to-one correspondence between one's risk preferences and her preferences over info-acts, and our definition of pessimism becomes behaviorally meaningful. So long as we speak relative to a given viable class, we can talk rigorously about the pessimism of a given risk preferences. And, fortunately, there are many viable classes, and some of these are quite rich in content. For instance, as we show in the body of the paper, if we focus on agents who compare two info-acts of the form (μ, f) and (μ, g) according to, say, maxmin, or maxmax, or Choquet, preferences, or a convex combination of these (as in the α-maxmin model), or more generally, according to what is called biseparable expected utility, then we obtain a viable class (Proposition 3). Similarly, the class of c-neutral preferences, that is, the collection of monotone preferences over info-acts which evaluate constant acts in a risk neutral manner, is viable (Proposition 4).

We demonstrate the usefulness of this approach toward pessimism by means of concrete examples. First, we show that our definition (relative to the viable class of biseparable preferences) agrees completely with the way literature defines pessimism in the context of RDU model (Proposition 5). Second, we show that this definition is significantly more refined than the latter definition. For instance, unlike the definition of pessimism in terms of probability weighting functions, we show in Proposition 6 that our definition of pessimism corresponds precisely to disappointment aversion in the context of Gul's model, sitting square with the typical way this model is interpreted. In a related application, we show in Proposition 7 that a cautious expected utility preference (cf. Cerreia-Vioglio, Dillenberger and Ortoleva (2013)) is pessimistic (relative to the viable class of c-neutral preferences) if and only if it is risk averse. The definition of pessimism through non-expected utility theory does not even apply to this model in a natural manner.

Intuitively, a "pessimistic person" would be particularly nervous about passing an opportunity of spreading the risk in her investments, in a way that may go beyond what risk aversion would account for. Indeed, an RDU preference that is pessimistic according to the standard definition would always look to diversifying her portfolio, *even if her utility for money is a linear function*. Unfortunately, this observation does not extend beyond this model. For instance, if pessimism is defined through weight distortions in a similar fashion in Gul's model, then it would not imply preference for diversification. By contrast, the final result of the present paper shows that the notion of pessimism we have introduced here (with respect to c-neutral preferences over info-acts) is generally consistent with intuition in this regard. Loosely speaking, we prove in Theorem 11 that, with our definition, every pessimistic person (with linear utility for money) exhibits preference for diversification. This property accords well with what one would intuitively expect from the behavior of a "pessimistic person," and hence provides further support for the general formulation we advance here.

2 Formulation of Risk as Uncertainty

The purpose of this section is to describe a model that modifies the standard framework of preferences over monetary lotteries to allow for *subjective* evaluation of "objective risk."

2.1 Nomenclature

We begin by introducing the order-theoretic nomenclature that is used in what follows. For any nonempty set S, by a **preference relation** $\succsim$ on S we mean a reflexive and transitive binary relation on S. The asymmetric (strict) part of $\succsim$ is denoted as $\succ$, and its symmetric (indifference) part is denoted as $\sim$. For any nonempty subset T of S, by the **restriction** of $\succsim$ to T, we mean the binary relation $\succsim |T := \succsim \cap (T \times T)$. We say that $\succsim$ is **complete** to mean that either $s \succsim t$ or $t \succsim s$ holds for every $s, t \in S$. A real function V on S is said to **represent** $\succsim$ if it is a utility function for $\succsim$, that is, $s \succsim t$ iff $V(s) \geq V(t)$, for every $s, t \in S$. When S is a topological space, we say that a preference relation $\succsim$ on S is **continuous** if it is a closed subset of $S \times S$ (relative to the product topology).

2.2 The State Space induced by Lotteries

The Lottery Space. We focus throughout this paper on monetary lotteries and/or acts with monetary payoffs. Consequently, even though parts of the discussion below apply in a more general context, we shall designate here a nondegenerate compact interval X in $\mathbb{R}$ as the *outcome space* of the model. By a **simple lottery** on X, we mean a Borel probability measure on X with finite support. The expectation of a continuous real map v on X with respect to a simple lottery $\mathbf{p}$ on X is denoted by $\mathbb{E}(v, \mathbf{p})$, but we write $\mathbb{E}(\mathbf{p})$ when v is the identity map on X.

The set of all simple lotteries on X is denoted as $\triangle(X)$. As usual, we view $\triangle(X)$ as a topological space relative to the topology of weak convergence: A net $(\mathbf{p}_\alpha)$ in $\triangle(X)$ converges to a simple lottery $\mathbf{p}$ on X iff $\mathbb{E}(v, \mathbf{p}_\alpha) \to \mathbb{E}(v, \mathbf{p})$ for every continuous real map v on X. Finally, the degenerate Borel probability measure on X that yields x with probability one is denoted as δ_x; we refer to such a measure as a **degenerate lottery**.

Preferences over Lotteries. We consider here only the complete and continuous preference relations on $\triangle(X)$. In addition, we always presume that money is a desirable commodity, so we posit at the outset that $\delta_x \succsim \delta_y$ iff $x \geq y$, for every $x, y \in X$. The set of all such preference relations on $\triangle(X)$ is denoted by $\mathfrak{R}(X)$.

The State Space. To describe the "subjective evaluation" of lotteries that we propose, the first step is designating a suitable state space. We will use for this purpose the set of all functions from $\triangle(X)$ into X that map each lottery to an element of its support. That is, we designate as our *state space* the set

$$\Omega := \left\{ \omega \in X^{\triangle(X)} : \omega(\mathbf{p}) \in \operatorname{supp}(\mathbf{p}) \text{ for each } \mathbf{p} \in \triangle(X) \right\}.$$

In words, Ω consists of the descriptions of all contingencies that may result when a person plays *any* one simple lottery on X. Put differently, a state ω in Ω describes what precisely happens in every lottery in the world, and hence, in concert with the Savagean modeling of uncertainty, represents the complete resolution of uncertainty.[4]

For any given lottery $\mathbf{p} \in \triangle(X)$ and $x \in \operatorname{supp}(\mathbf{p})$, we let

$$\Omega(\mathbf{p}, x) := \{\omega \in \Omega : \omega(\mathbf{p}) = x\},$$

which is the event that "the outcome x obtains in the lottery $\mathbf{p}$." Then, $\Omega(\mathbf{p}) := \{\Omega(\mathbf{p}, x) : x \in \operatorname{supp}(\mathbf{p})\}$ is a (finite) partition of the state space Ω that corresponds to the uncertainty embodied in $\mathbf{p}$.

Info-Acts. Suppose $\mathbf{p}$ is the lottery that corresponds to the tossing of a fair coin, one that pays \$10 if Heads come up and $-$\$5 otherwise. As we have noted in the Introduction, we propose to view this lottery as an act that pays a sure payoff of \$10 in the event "Heads come up in the toss," and $-$\$5 in the event "Tails come up in the toss," together with the information that these events are equally likely. Assuming that -5 and 10 belong to X, our formulation captures the event "Heads come up in the toss," by $\Omega(\mathbf{p}, 10)$, and the event "Tails come up in the toss," by $\Omega(\mathbf{p}, -5)$. Thus, we let an agent conceive $\mathbf{p}$ as a (Savagean) *act* on Ω, say, f, one that pays \$10 at any state in $\Omega(\mathbf{p}, 10)$, and $-$\$5 at any other state in Ω. The agent is, however, aware that these events are equally likely, that is, she is given a probability distribution over the partition $\Omega(\mathbf{p})$ that assigns probability $\frac{1}{2}$ to the event $\Omega(\mathbf{p}, 10)$. Denoting this particular distribution by μ, therefore, we model the objective lottery $\mathbf{p}$ as the pair (μ, f), thereby allowing the decision maker to look at things subjectively.

We now formalize this idea. For any $\mathbf{p}$ in $\triangle(X)$, let us denote by $\langle \mathbf{p} \rangle$ the probability measure on the algebra generated by $\Omega(\mathbf{p})$ on Ω such that $\langle \mathbf{p} \rangle(\Omega(\mathbf{p}, x)) = \mathbf{p}(x)$ for each x

[4]Once X is specified, Ω is uniquely defined. To simplify the notation, however, we do not use a notation that makes the dependence of Ω on X explicit.

in $\mathrm{supp}(\mathbf{p})$. In turn, we denote by $\mathcal{F}(\mathbf{p})$, the set of all $\triangle(X)$-valued acts on $\mathcal{F}(\mathbf{p})$, that is, the set of all maps $f : \Omega \to \triangle(X)$ that are measurable with respect to this algebra. Put differently, $f \in \mathcal{F}(\mathbf{p})$ iff f is a map from Ω into $\triangle(X)$ that is constant on each element of the partition $\Omega(\mathbf{p})$. If every value of $f \in \mathcal{F}(\mathbf{p})$ is a degenerate lottery, that is, $f(\Omega)$ is contained in $\{\delta_x : x \in X\}$, we say that f is **degenerate-valued.** A particularly important degenerate-valued act in $\mathcal{F}(\mathbf{p})$ is the one that pays exactly what $\mathbf{p}$ pays on the event that "the outcome x obtains in the lottery $\mathbf{p}$." Throughout this paper, we denote this act by $f_{\mathbf{p}}$: Put explicitly, $f_{\mathbf{p}}$ is the act on Ω such that $f|_{\Omega(\mathbf{p},\mathbf{x})} = \delta_x$ for every $x \in \mathrm{supp}(\mathbf{p})$, that is,

$$f_{\mathbf{p}} = \sum_{x \in \mathrm{supp}(\mathbf{p})} \delta_x \mathbf{1}_{\Omega(\mathbf{p},x)}.$$

Convention. For any $\mathbf{p} \in \triangle(X)$, $f \in \mathcal{F}(\mathbf{p})$ and $x \in \mathrm{supp}(\mathbf{p})$, we denote "the" value of f on $\Omega(\mathbf{p}, x)$ by $f(x)$ throughout the exposition. That is, we set $f(x) := f(\omega)$ for some (and hence any) ω in $\Omega(\mathbf{p}, x)$.[5]

By an **info-act** on Ω, we mean an ordered pair $(\langle \mathbf{p} \rangle, f)$ where $\mathbf{p}$ is a simple lottery on X and $f \in \mathcal{F}(\mathbf{p})$. We denote the set of all info-acts by $\blacktriangle(X)$, that is,

$$\blacktriangle(X) := \{(\langle \mathbf{p} \rangle, f) : \mathbf{p} \in \triangle(X) \text{ and } f \in \mathcal{F}(\mathbf{p})\}.$$

A **degenerate info-act** on Ω is a member $(\langle \mathbf{p} \rangle, f)$ of $\blacktriangle(X)$ such that f is a degenerate-valued act on $\mathcal{F}(\mathbf{p})$. A particularly important such info-act is $(\langle \mathbf{p} \rangle, f_{\mathbf{p}})$, which we refer to as the **info-act** on Ω **induced by p**. The idea is that, when evaluating a lottery $\mathbf{p}$, which is a risky prospect, the agent feels like she actually faces an uncertain prospect. She knows that this prospect will pay her x in the "event $\Omega(\mathbf{p}, x)$," as well as the fact that the "event $\Omega(\mathbf{p}, x)$" is declared to obtain with probability $\mathbf{p}(x)$. So, the image of the lottery $\mathbf{p}$ in her mind is precisely the info-act $(\langle \mathbf{p} \rangle, f_{\mathbf{p}})$.

There are various ways of turning $\blacktriangle(X)$ into a topological space. We do this by endowing $\blacktriangle(X)$ with a topology that is intuitively consistent with the weak convergence of probability measures: We say that a net (μ_α, f_α) in $\blacktriangle(X)$ converges to an info-act (μ, f) on Ω iff

$$\int_\Omega \mathbb{E}(v, f_\alpha(\cdot))\, \mathrm{d}\mu_\alpha \to \int_\Omega \mathbb{E}(v, f(\cdot))\, \mathrm{d}\mu$$

for every continuous real map v on X.

Before we proceed further, let us point to a caveat. Notice that the codomain of f in any info-act (μ, f) is $\triangle(X)$, instead of X, that is, "acts" are modeled here as the Anscombe-Aumann horse-race lotteries, as opposed to Savagean acts. While taking a purely Savagean approach would intuitively be more appropriate for the present exercise, that approach is marred with considerable technical complications. This is, of course, typical in decision theory. Indeed, a vast majority of the recent developments in the theory of decision making under uncertainty exploits the richer structure of the Anscombe-Aumann framework instead of the

[5] In other words, we define $\hat{f} : \mathrm{supp}(\mathbf{p}) \to \Delta(X)$ by $\hat{f}(x) := f(\omega)$ for some (and hence any) ω in $\Omega(\mathbf{p}, x)$, and as a convention, identify f with $\hat{f}$.

Savagean one. For tractability reasons on one hand, and because we will make subsequent use of these developments on the other, we also adopt this framework in the present paper.

Interpretation of Info-Acts. $\blacktriangle(X)$ provides a setting for talking about *subjective risk*. The observable framework is the standard one in which a decision-maker ranks monetary lotteries. The upshot here is that the decision-maker need not take the probabilities in a given lottery as declared. Instead, she uses these probabilities as information about the likelihood of the outcomes of the lottery, and may form beliefs about these likelihoods that differ from what is prescribed by the lottery. To wit, if $\mathbf{p}$ is a simple lottery on X, then the agent actually views this lottery as the info-act $(\langle \mathbf{p} \rangle, f)$, where $\langle \mathbf{p} \rangle(\Omega(\mathbf{p}, x)) = \mathbf{p}(x)$ and $f(\omega) = \delta_x$ for every $\omega \in \Omega$ with $\omega(\mathbf{p}) = x$.

Preferences over Info-acts. We consider here only the complete and continuous preference relations on $\blacktriangle(X)$. Besides, we presume that a person with a preference relation $\succsim$ on $\blacktriangle(X)$ prefers more money to less regardless of the information she is given about the likelihood of various states, that is, we posit at the outset that $(\langle \mathbf{p} \rangle, \delta_x \mathbf{1}_\Omega) \succsim (\langle \mathbf{p} \rangle, \delta_y \mathbf{1}_\Omega)$ iff $x \geq y$, for every $x, y \in X$ and $\mathbf{p} \in \triangle(X)$. (This property is referred to as the "desirability of money" in what follows.)[6]

State-Invariance. The set of all info-acts on Ω contains some redundancies. That is, it contains info-acts that one cannot distinguish from each other behaviorally. This necessitates imposing certain invariance properties on the preferences over info-acts. For instance, consider a lottery $\mathbf{p}$ that pays \$1 with probability $\frac{1}{6}$, \$2 with probability $\frac{1}{3}$, and \$3 with probability $\frac{1}{2}$, and an act f in $\mathcal{F}(\mathbf{p})$ which pays a lottery a_i in the event $\Omega(\mathbf{p}, i)$, $i = 1, 2, 3$. Now compare this with the info-act $(\langle \mathbf{q} \rangle, g)$, where $\mathbf{q}$ is the lottery that pays x_1 with probability $\frac{1}{6}$, x_2 with probability $\frac{1}{3}$, and x_3 with probability $\frac{1}{2}$, and g is that act in $\mathcal{F}(\mathbf{q})$ which pays a_i in the event that the outcome x_i obtains in the lottery $\mathbf{q}$. It is clear that the difference between the info-acts $(\langle \mathbf{p} \rangle, f)$ and $(\langle \mathbf{q} \rangle, g)$ is immaterial. Indeed, these info-acts partition the state space into three events, say, $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$, respectively, and inform the decision-maker that the probabilities of A_i and B_i are the same for each $i = 1, 2, 3$. (Here, $A_i = \Omega(\mathbf{p}, i)$ and $B_i = \Omega(\mathbf{q}, x_i)$ for each i.) Furthermore, the payoff of $(\langle \mathbf{p} \rangle, f)$ in the event A_i and that of $(\langle \mathbf{q} \rangle, g)$ in the event B_i are the same, namely, a_i, for each i. Clearly, while the formalism of the model treats $(\langle \mathbf{p} \rangle, f)$ and $(\langle \mathbf{q} \rangle, g)$ as different objects, the interpretation of it would cease to make sense if we allowed an individual perceive these info-acts as such. We must, therefore, impose on a preference relation $\succsim$ a "state anonymity" condition that says that preferences are invariant under the renaming of states, thereby ensuring that $(\langle \mathbf{p} \rangle, f) \sim (\langle \mathbf{q} \rangle, g)$. (See Figure 2.)

There is another situation that requires imposing a similar invariance condition. Indeed, our interpretation requires an agent not distinguish between two info-acts (μ, f) and (ν, f) so long as f is a constant act. More generally, we should make sure that the given probabilities

[6] Insofar as the primitives of the model are lotteries, and hence info-acts are the analyst's constructs, we cannot determine one's preferences over non-degenerate info-acts from her risk preferences. It is in this sense that preferences over info-acts are not observable. Having said this, info-acts are concrete objects, and we can certainly determine an individual's preferences over them by asking her to compare such objects (say, in experimental settings). Thus, potentially, albeit, not from the ranking of lotteries, preferences over info-acts are observable entities.

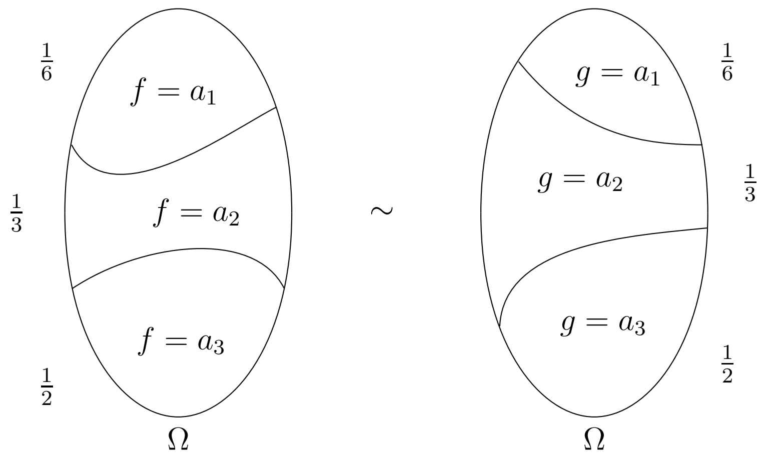

Figure 2: The State Anonymity Property

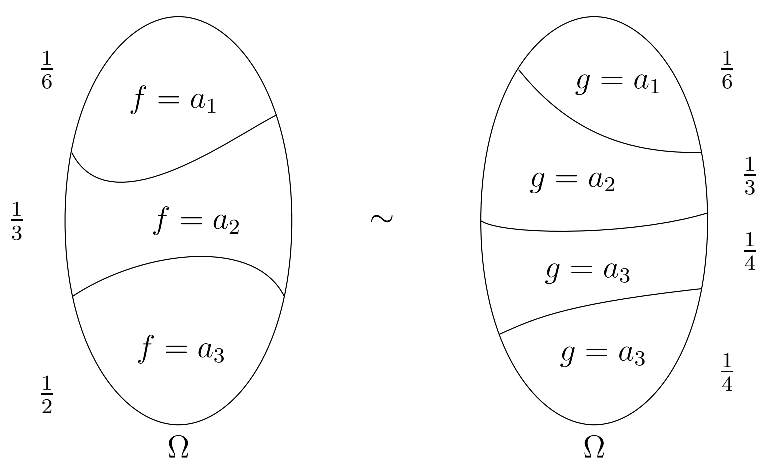

Figure 3: The Info-Irrelevance Property

of any two states are immaterial for preferences in evaluating an act that yields the same outcome on those two states (other than what they entail for the likelihood of those two states put together). To illustrate, let $(\langle \mathbf{p} \rangle, f)$ be as above and consider the info-act $(\langle \mathbf{q} \rangle, g)$ where $\Omega(\mathbf{q}) = \{B_1, ..., B_4\}$ and $\langle \mathbf{q} \rangle$ assigns probability $\frac{1}{6}$ to B_1, $\frac{1}{3}$ to B_2, $\frac{1}{4}$ to B_3, and $\frac{1}{4}$ to B_4, while g pays a_i in the event B_i, $i = 1, 2$, and a_3 in the event $B_3 \cup B_4$. It is clear that the difference between $(\langle \mathbf{p} \rangle, f)$ and $(\langle \mathbf{q} \rangle, g)$ is immaterial for choice, so we must impose an invariance condition that ensures that $(\langle \mathbf{p} \rangle, f) \sim (\langle \mathbf{q} \rangle, g)$. (See Figure 3.)

We thus introduce the following property on one's preferences over info-acts that provides a joint formalization of these two invariance properties.

The State Invariance Axiom. Let $(\langle \mathbf{p} \rangle, f)$ and $(\langle \mathbf{q} \rangle, g)$ be two info-acts on Ω for which there is a map $\sigma : \Omega(\mathbf{q}) \to \Omega(\mathbf{p})$ such that

(i) $f(\omega') = g(\omega)$ for every $S \in \Omega(\mathbf{q})$ and $(\omega, \omega') \in S \times \sigma(S)$; and

(ii) $\langle \mathbf{q} \rangle(\sigma^{-1}(T)) = \langle \mathbf{p} \rangle(T)$ for every $T \in \sigma(\Omega(\mathbf{q}))$.

Then, it must be that $(\langle \mathbf{p} \rangle, f) \sim (\langle \mathbf{q} \rangle, g)$.

The set of all complete and continuous preference relations on $\blacktriangle(X)$ that satisfy the desirability of money and State Invariance Axioms is denoted by $\mathfrak{P}(X)$.

Related Models. To the best of our knowledge, preferences over prior-act pairs (that is, info-acts) have not been considered in the literature. Work that comes closest to doing this pertain to ranking ordered pairs such as $(\mathcal{M}, f)$ where $\emptyset \neq \mathcal{M} \subseteq \triangle(\Omega)$ and $f \in X^\Omega$; such preferences are studied by Gajdos, Tallon and Vergnaud (2004) and Gajdos, Hayashi, Tallon and Vergnaud (2008) in the Anscombe-Aumann setup, among others. By way of interpretation, f is viewed as a typical act in that literature while $\mathcal{M}$ models imprecise, but objective, information about the likelihoods of states of nature. This may at first suggest that our setup is a special case of that work, one in which $\mathcal{M}$s are singletons. This is, however, not the case. Indeed, preferences in that literature are, without exception, such that the information content of a set $\mathcal{M}$ is never questioned. In particular, the utility of a pair $(\{\mu\}, f)$ is precisely the expected utility of the act f with respect to the prior μ. Thus, such preferences cannot be used to study "subjective risk" as we attempt to do here.

The only studies (that we are aware of) in which uncertainty arises in an objective setting are Olszewski (2007) and Ahn (2008). But these works are about preferences over sets of lotteries, and are dynamic in essence. Again, when ranking singletons of lotteries, these theories collapse to the expected utility paradigm – as such, they are not suitable for studying situations in which one evaluates a given objective lottery subjectively.

2.3 Mapping Lotteries to Info-Acts

The Canonical Map. In what follows, we refer to the function that maps any given lottery $\mathbf{p}$ in $\triangle(X)$ to the info-act on Ω induced by $\mathbf{p}$ as the **canonical map**, and denote it by φ. This map specifies precisely the way in which the decision-maker transforms in her mind a given lottery into an info-act. Formally, we define the map $\varphi : \triangle(X) \to \blacktriangle_{\mathrm{d}}(X)$ by

$$\varphi(\mathbf{p}) := (\langle \mathbf{p} \rangle, f_{\mathbf{p}}), \tag{1}$$

which is easily checked to be continuous (where we view $\blacktriangle_{\mathrm{d}}(X)$ as a topological subspace of $\blacktriangle(X)$). To reiterate, the interpretation is that the monetary lottery $\mathbf{p}$ (which is meant to be evaluated objectively) becomes the info-act $\varphi(\mathbf{p})$ in the mind of the decision maker (which is evaluated subjectively).

Mapping Preferences over Lotteries to Preferences over Info-Acts. The canonical map φ provides a way of reading the preferences of a decision-maker over monetary lotteries (i.e., her (observable) risk preferences) in terms of her preferences over info-acts (i.e., her (partially observable) subjective risk preferences). To formalize this, we define the map Φ from $\mathfrak{P}(X)$ into the set of all binary relations on $\triangle(X)$ as follows:

$$\mathbf{p}\,\Phi(\succsim)\,\mathbf{q} \quad \text{iff} \quad \varphi(\mathbf{p}) \succsim \varphi(\mathbf{q})$$

for any $\mathbf{p}, \mathbf{q} \in \triangle(X)$. Thus, if the preferences of the individual over info-acts are given by $\succsim$, we understand that she would prefer a monetary lottery $\mathbf{p}$ over another one, say, $\mathbf{q}$ – that is, $\mathbf{p}\,\Phi(\succsim)\,\mathbf{q}$ – if, and only if, she prefers the info-act formulation of $\mathbf{p}$ to that of $\mathbf{q}$, that is, $\varphi(\mathbf{p}) \succsim \varphi(\mathbf{q})$.

A few preliminary observations about the map Φ are in order. For any given $\succsim$ in $\mathfrak{P}(X)$, it is obvious that $\Phi(\succsim)$ is a complete preference relation on $\triangle(X)$, while continuity of φ entails that $\Phi(\succsim)$ is a closed subset of $\triangle(X) \times \triangle(X)$. Besides, using the properties of State Invariance and desirability of money (for $\succsim$), we see that $\delta_x\ \Phi(\succsim)\ \delta_y$ iff $x \geq y$, for every $x, y \in X$.[7] Thus, $\Phi(\mathfrak{P}(X)) \subseteq \mathfrak{R}(X)$, so we may, and will, treat Φ as a function from $\mathfrak{P}(X)$ into $\mathfrak{R}(X)$.

Looking Ahead. The map Φ provides a concrete pathway towards reading off the consequences of one's attitudes toward uncertainty for her attitudes in terms of risk. In particular, we can posit conditions on how an agent evaluates info-acts, and then using Φ we can deduce what sort of risk preferences such an agent would have. This allows us to relate the theory of choice under uncertainty to decision theory under risk in a nontrivial manner. To put this more concretely, let us define, for any preference relation $\succsim$ in $\mathfrak{P}(X)$ and $\mathbf{p}$ in $\triangle(X)$, the preference relation $\succsim^{\mathbf{p}}$ on $\mathcal{F}(\mathbf{p})$ as follows:

$$f \succsim^{\mathbf{p}} g \quad \text{iff} \quad (\langle \mathbf{p} \rangle, f) \succsim (\langle \mathbf{p} \rangle, g). \tag{2}$$

Notice that $\succsim^{\mathbf{p}}$ is in essence a preference relation over certain types of acts on Ω, and as such, it is precisely the primitive of the theory of choice under uncertainty. Therefore, we can take any one particular property that is found of use in decision theory under uncertainty, or one that commands support through experiments concerning uncertain prospects, and assume that $\succsim^{\mathbf{p}}$ has that property for *all* $\mathbf{p}$ in $\triangle(X)$. As a consequence, we may then identify what the risk preferences of the agent, namely, $\Phi(\succsim)$, would then look like. In particular, we shall show in the next section that rather basic conditions on one's preferences $\succsim$ over info-acts would ensure that $\Phi(\succsim)$ must abide by a well-known type of non-expected utility theory.

Monotonicity of Preferences over Info-Acts. In what follows, we shall make abundant use of a standard axiom of the theory of choice under uncertainty, namely *monotonicity*. As noted

[7]For any x and y in X, we have $x \geq y$ iff $(\langle \delta_x \rangle, \delta_x \mathbf{1}_\Omega) \succsim (\langle \delta_x \rangle, \delta_y \mathbf{1}_\Omega)$ by desirability of money, and the latter statement holds iff $(\langle \delta_x \rangle, \delta_x \mathbf{1}_\Omega) \succsim (\langle \delta_y \rangle, \delta_y \mathbf{1}_\Omega)$, that is, $\varphi(\delta_x) \succsim \varphi(\delta_y)$, by the State Invariance Axiom.

above, this property is readily adopted to the present framework through imposing it on the relation $\succsim^{\mathbf{p}}$ for each $\mathbf{p}$. In particular, we say that a preference relation $\succsim$ in $\mathfrak{P}(X)$ is **monotonic** if so is $\succsim^{\mathbf{p}}$ on $\mathcal{F}(\mathbf{p})$ for each $\mathbf{p} \in \triangle(X)$, which means that, for every $\mathbf{p} \in \triangle(X)$ and $f, g \in \mathcal{F}(\mathbf{p})$, we have $f \succsim^{\mathbf{p}} g$ whenever

$$f(x)\mathbf{1}_\Omega \succsim^{\mathbf{p}} g(x)\mathbf{1}_\Omega \qquad \text{for each } x \in \operatorname{supp}(\mathbf{p}). \tag{3}$$

2.4 The Subjective Risk Model vs. Non-Expected Utility Theory

The General Weighted EU Model. The most prominent of the non-expected utility theories to date provides a representation of preference relations in $\mathfrak{R}(X)$ through utility functions $U : \triangle(X) \to \mathbb{R}$ of the form

$$U(\mathbf{p}) := \sum_{x \in \operatorname{supp}(\mathbf{p})} \pi(x, \mathbf{p}) u(x) \tag{4}$$

where $u : X \to \mathbb{R}$ is a continuous and strictly increasing function and π is an $\mathbb{R}_+$-valued map on $\{(x, \mathbf{p}) : \mathbf{p} \in \triangle(X) \text{ and } x \in \operatorname{supp}(\mathbf{p})\}$ such that $\sum_{x \in \operatorname{supp}(\mathbf{p})} \pi(x, \mathbf{p}) = 1$ for every $\mathbf{p} \in \triangle(X)$. Here π is referred to as a **probability weighting function** on X, and the representation is called the **general weighted EU model**. (We refer to a preference relation on $\triangle(X)$ that is represented by such a function as a **general weighted EU preference which is represented by** (π, u).) This model contains several non-expected utility models, such as the weighted expected utility theory (Chew and MacCrimmon (1979) and Chew (1983)), the dual expected utility model (Yaari (1987)), the rank-dependent utility model (Quiggin (1982) and Chew (1983)), the cumulative prospect theory (Tversky and Kahneman (1992) and Chateauneuf and Wakker (1999)), and the theory of disappointment aversion (Gul (1991)), among others.[8] The interpretation of this general representation is that it is "as if" the decision maker distorts the objectively given probabilities in a lottery, and then makes her evaluation by using these distorted probabilities to compute the expected utility of that lottery.

In the literature on non-expected utility theory, utility functions that are of specializations of the form (4) are characterized by imposing conditions (that weaken the von Neumann-Morgenstern independence axiom) on an arbitrarily given preference relation $\trianglerighteq$ in $\mathfrak{R}(X)$. As such, "distortions of probabilities" are obtained in the representation theorems mainly as a by-product, but one that nevertheless enjoys a useful interpretation. It is indeed tempting to interpret an agent with utility function (4) as "believing that x will obtain in the lottery $\mathbf{p}$ with probability $\pi(x, \mathbf{p})$." However, this interpretation is markedly different than how beliefs arise in the theory of choice under uncertainty. In particular, due to the lack of a suitable state space, it is not even clear how to think of the notion of "beliefs" in the context of risk. In particular, unlike the situation in the Savagean theory, it is not possible in this context to elicit one's beliefs about various events by asking her to evaluate bets on them. As a result, the interpretation of $\pi(x, \mathbf{p})$ as one's belief that x will obtain in $\mathbf{p}$ does not have a choice-theoretic foundation. In fact, the primary goal of our subjective risk model is precisely to provide such a foundation.

[8] See Starmer (2000), Sugden (2004), Schmidt (2004) and Machina (2008) for insightful surveys on non-expected utility theory in the context of risk.

On the Structure of Risk Preferences induced by Φ*.* The subjective risk theory we have introduced above draws a contrast in the treatment of the distortion of objectively given probabilities. This theory aims to model the phrase "as if one distorts the given probabilities in her mind" by allowing a person to view a lottery as an info-act (in her mind), thereby making room for the agent to view an objective lottery *subjectively* by transforming it (through the canonical map φ) into an info-act. This allows considering hypotheses about one's attitudes toward uncertain prospects (info-acts), and hence, imposing structure to her preferences in this mental realm. The map Φ then lets us transform these preferences back into the world of (observable) preferences over risky prospects. As such, this theory attempts to provide foundations for the very phenomenon that the general weighted EU model captures in the form of an "as if" interpretation.

We now show that this attempt is indeed successful. For, under a very weak monotonicity hypothesis, every preference relation over info-acts yields (through Φ) a risk preference that carries the structure of the general weighted EU model. This is the content of the following observation.

Proposition 1. *Let* $\succsim$ *be a preference relation in* $\mathfrak{P}(X)$ *such that*

$$\delta_{\max \operatorname{supp}(\mathbf{p})}\mathbf{1}_\Omega \succsim^{\mathbf{p}} f_{\mathbf{p}} \succsim^{\mathbf{p}} \delta_{\min \operatorname{supp}(\mathbf{p})}\mathbf{1}_\Omega \quad \text{for every } \mathbf{p} \in \triangle(X). \tag{5}$$

Then, $\Phi(\succsim)$ *can be represented as in the general weighted EU model.*

In particular, for every monotonic preference relation $\succsim$ over info-acts, $\Phi(\succsim)$ is sure to be captured by the general weighted EU model, provided that the individual finds money desirable. Indeed, the condition (5) corresponds to a notion which is much less demanding than consistency with first-order stochastic dominance. Take any simple lottery $\mathbf{p}$ on X, which the agent interprets as the "info." Then, $\delta_{\max \operatorname{supp}(\mathbf{p})}\mathbf{1}_\Omega$ is just another label for the lottery that pays the best outcome in the support of $\mathbf{p}$ while, of course, $f_{\mathbf{p}}$, the info-act induced by $\mathbf{p}$, is just another label for the lottery $\mathbf{p}$ itself. Consequently, it is natural to posit that the former would be deemed better than the latter by the agent, and this is precisely what the first part of (5) says, while its second part is understood analogously. As such, (5) is hardly exceptionable, and thus Proposition 1 can be thought of saying that all reasonable preferences over-info acts induce risk preferences that fall within the general weighted EU model. Put differently, subjectivity we allow for in the world of info-acts manifests itself in the realm of risky prospects as distortion of probabilities, which are then used for the computation of expected utilities. The intuitive subjectivity of the latter is captured explicitly in this approach.

3 Subjective Risk and Ambiguity

There are no behavioral justifications for probability distortions within the confines of non-expected utility theory. This theory is based on consistency properties that help characterize preferences that could be represented by a utility function in which some type of probability weighting occurs. But, precisely because it does not model the potentially subjective evaluation of the likelihoods in a given lottery at the level of primitives, it provides limited insight about the underlying behavioral reasons behind such distortions.

By contrast, the approach of transforming a given risk preference into a subjective risk preference seems more promising in this regard. As we shall demonstrate in this section, this approach shows that, under very general circumstances, whether or not a decision maker distorts objective probabilities (and then use the distorted probabilities to make expected utility computations) depends vitally on her attitude toward ambiguity. This is the main theoretical finding of this paper: such distortions may occur *only if* the agent is not neutral toward ambiguity (in the realm of info-acts). As such, the subjective risk theory identifies a somewhat unexpected connection between one's "attitudes toward ambiguity" and her "evaluation of risky prospects." At the very least, this provides a novel behavioral viewpoint about the decision maker who appears as if she evaluates lotteries by their expected utilities with distorted probabilities.

Ambiguity Neutrality. Let $\succsim_1$ and $\succsim_2$ be two preference relations (over info-acts) in $\mathfrak{P}(X)$. Following Ghirardato and Marinacci (2002), we say that $\succsim_1$ is **at least as ambiguity averse** as $\succsim_2$ whenever

$$(\langle \mathbf{p} \rangle, r\mathbf{1}_\Omega) \succsim_2 (\langle \mathbf{p} \rangle, f) \quad \text{implies} \quad (\langle \mathbf{p} \rangle, r\mathbf{1}_\Omega) \succsim_1 (\langle \mathbf{p} \rangle, f) \tag{6}$$

for every $r \in \triangle(X)$ and $(\langle \mathbf{p} \rangle, f)$ in $\blacktriangle(X)$. The idea is simply that if a person prefers a constant info-act to another (possibly non-constant) info-act, but both with the same "info," then a more ambiguity averse person would surely do the same. If $\succsim_1$ and $\succsim_2$ are at least as ambiguity averse as each other, we say that they are **equally ambiguity averse**. Finally, if $\succsim_1$ and $\succsim_2$ are equally ambiguity averse, and if, for each $\mathbf{p}$ in $\triangle(X)$, the preference relation $\succsim_2^{\mathbf{p}}$ (as defined in (2)) has an Ancombe-Aumann expected utility representation on $\mathcal{F}(\mathbf{p})$, we say that $\succsim_1$ is **ambiguity neutral**. In particular, if a preference relation $\succsim$ on $\blacktriangle(X)$ can be represented by a utility function $U : \blacktriangle(X) \to \mathbb{R}$ of the form

$$U(\langle \mathbf{p} \rangle, f) := \sum_{x \in \text{supp}(\mathbf{p})} \mu_{\mathbf{p}}(\Omega(\mathbf{p}, x))\mathbb{E}(u, f(x)) \tag{7}$$

for some continuous and strictly increasing $u : X \to \mathbb{R}$ and some self-map $\mathbf{p} \mapsto \mu_{\mathbf{p}}$ on $\triangle(\Omega)$, then $\succsim$ must be ambiguity neutral.

Ambiguity Neutrality implies Undistorted Probabilities. Informally speaking, our main assertion in this paper can be stated as follows. The choice behavior of an individual would deviate from expected utility theory in the context of risk provided that:

(1) the decision-maker views any given lottery as an info-act; and

(2) the preference relation of the decision-maker over info-acts is not ambiguity neutral.

It turns out that this contention can be demonstrated in great generality. Given the standard monotonicity hypothesis, if the preferences of an individual over info-acts are ambiguity neutral, then the induced preferences (by Φ) over risky prospects can be represented as in (7) *with beliefs matching the prior of any info-act*, that is, with $\mu_{\mathbf{p}} = \langle \mathbf{p} \rangle$ for each $\mathbf{p} \in \triangle(X)$. Put precisely:

Theorem 2. *Let $\succsim$ be a monotonic element of $\mathfrak{P}(X)$. If $\succsim$ is ambiguity neutral, then there is a continuous and strictly increasing $u : X \to \mathbb{R}$ such that*

$$\mathbf{p}\, \Phi(\succsim)\, \mathbf{q} \quad \text{iff} \quad \mathbb{E}(u, \mathbf{p}) \geq \mathbb{E}(u, \mathbf{q})$$

for every $\mathbf{p}, \mathbf{q} \in \triangle(X)$.[9]

In words, a decision maker whose risk preferences arise from her (monotonic) preferences over info-acts would not at all distort the probabilities given in a lottery, and behave as a standard expected utility maximizer, provided that she is neutral to ambiguity when comparing info-acts. We may thus argue that the failure of the von Neumann-Morgenstern independence axiom in general (as in the Allais paradox), and the apparent distortion of probabilities of risky prospects in particular, are intimately linked to the non-neutrality of one's attitudes toward ambiguity (as in the two-urn Ellsberg paradox). This seems to provide a novel perspective to the notion of objective probability distortions. In particular, it gives grounds for the somewhat unusual claim that Allais' and related paradoxes in risky environments are but certain manifestations of the Ellsberg type phenomena. It does not explain the "cause" of why one distorts probabilities in the world of risk, from neither a psychological nor evolutionary standpoint, but it shows that this cause must be searched within the "causes" of why one may not be neutral toward ambiguity.

In passing, we should emphasize that Theorem 2 is not an idle theoretical observation. It shows that one may be able to identify the structure of the way a person may distort objective probabilities from the manner in which she evaluates uncertainty. In particular, this paves the way towards studying "pessimistic" or "optimistic" ways of distorting probabilities in terms of one's attitude towards ambiguity. The rest of the paper focuses on this issue.

4 Subjective Risk and Pessimism

As it allows a decision maker to subjectively evaluate a lottery by distorting the likelihoods of payoffs that are given in an objective sense, the model of preferences over info-acts is primed to capture behavioral traits of pessimism and optimism. This section aims at demonstrating how this can be done. We will first introduce a theory of pessimism by using the subjective risk model of Section 2, and then revisit how the notion of pessimism is modeled in the literature on non-expected utility theory. We will find that our theory conforms with how pessimism is modeled in the literature in certain important contexts, but also that it delivers considerably more acceptable answers in others.

4.1 Pessimism in the Subjective Risk Model

A Pseudo-Definition of Pessimism. As important as it is from the behavioral perspective, pessimism is an elusive concept when it comes to defining it in the context of risky environments. Unlike, say, risk aversion, pessimism seems difficult to define purely choice theoretically. It seems that this concept is, at least in part, a psychological phenomenon, and this makes a revealed preference formulation of it untenable. It is thus not surprising that the literature

[9] A slightly more general statement than this is the following: A preference relation $\trianglerighteq \in \Phi(\mathfrak{P}(X))$ admits an expected utility representation with respect to a continuous and strictly increasing utility function if, and only if, there exists an ambiguity neutral $\succsim \in \mathfrak{P}(X)$ such that $\trianglerighteq = \Phi(\succsim)$. We also note that it is possible to formulate Theorem 2 in a Savagean context (in which only degenerate info-acts are used), for this result does not really depend on the structure of the agent's preferences over constant acts. For brevity, however, we do not provide this version of the result in this paper.

does not give such a definition at large, but instead, provides definition(s) in the context of special types of preferences through their functional (utility) representations. We will talk quite a bit about the difficulties of this approach in Section 4.3, but let us note at once that the situation draws a stark contrast with the theory of choice under uncertainty.

In the context of uncertainty where there is explicit room for subjective evaluation of the likelihoods of states, there is a largely uncontested, and a widely adopted, notion of "pessimism." This notion identifies pessimism with "preference for hedging," and formulates it by means of the famous *Uncertainty Aversion* axiom of Gilboa and Schmeidler (1989). And, of course, we can readily adopt this formulation to the context of preferences over info-acts: A preference relation $\succsim$ on $\blacktriangle(X)$ is **uncertainty averse** if $\succsim^{\mathbf{p}}$ satisfies this axiom, that is, for all $f, g \in \mathcal{F}(\mathbf{p})$ and $0 \leq \lambda \leq 1$,

$$(\langle \mathbf{p} \rangle, f) \sim (\langle \mathbf{p} \rangle, g) \quad \text{implies} \quad (\langle \mathbf{p} \rangle, \lambda f + (1-\lambda) g) \succsim (\langle \mathbf{p} \rangle, f),$$

for any given $\mathbf{p} \in \triangle(X)$. If $\succsim$ is replaced with $\sim$ in this statement, we obtain the definition of **uncertainty neutral** preferences over info-acts, and if it is replaced with $\precsim$, we obtain that of **uncertainty loving** preferences over info-acts.

Uncertainty aversion of $\succsim$ is meant to capture the "psyche" of the individual, and hence, it is in full concert with viewing one's pessimism as a psychological phenomenon. If one's risk preferences arise from uncertainty averse preferences over info-acts, therefore, it seems reasonable to categorize that person as being pessimistic at large. More precisely, we would like to say that a risk preference $\trianglerighteq$ on $\triangle(X)$ is *pessimistic* (*optimistic*) if this preference relation is the image of an uncertainty averse (loving) preference relation $\succsim$ on $\blacktriangle(X)$.

Unfortunately, while this appears to be a novel formulation, and is based on a promising intuition, it is a bit too good to be useful. After all, we may in general associate a given (observable) risk preference with a multitude of preference relations over info-acts. Indeed, the map Φ is not injective, it takes $\mathfrak{P}(X)$ into $\mathfrak{R}(X)$ in a many-to-one manner.[10] To see this, given any $\succsim$ in $\mathfrak{P}(X)$ and continuous self-map F on $\triangle(X)$, consider the preference relation $\succsim_F \in \mathfrak{P}(X)$ defined as

$$(\mu, f) \succsim_F (\nu, g) \quad \text{iff} \quad (\mu, F \circ f) \succsim (\nu, F \circ g).$$

Clearly, $\Phi(\succsim_F)$ is the same preference relation in $\mathfrak{R}(X)$ for *any* F such that $F(\delta_x) = \delta_x$ for each $x \in X$. Therefore, Φ is not injective.

Non-injectivity of the map Φ means that, in general, we cannot identify the preferences over info-acts by observing one's preferences over monetary lotteries. This, in turn, causes a severe difficulty for defining the pessimism of one's risk preferences as we have intuitively suggested above. As the following example illustrates, two different preferences over info-acts, one pessimistic (that is, uncertainty averse) and the other not, may induce through Φ the same risk preferences.

[10] This is exactly where we pay the price for modeling acts here in the tradition of Anscombe-Aumann as opposed to Savage. If we defined an info-act (μ, f) as one in which the values of f are elements of X (instead of $\Delta(X)$), this sort of an invertibility problem would not arise. However, as we have noted earlier, in that case we would encounter other sorts of difficulties, and would surely not be able to invoke the results we need from the recent literature on decision-making under uncertainty that takes the Anscombe-Aumann model as its primary setup.

Example 1. Consider the preference relation $\succsim$ on $\blacktriangle(X)$ defined as

$$(\langle \mathbf{p} \rangle, f) \succsim (\langle \mathbf{q} \rangle, g) \quad \text{iff} \quad \sum_{x \in \text{supp}(\mathbf{p})} \mathbf{p}(x)\mathbb{E}(f(x)) \geq \sum_{x \in \text{supp}(\mathbf{q})} \mathbf{q}(x)\mathbb{E}(g(x)).$$

Next, let v_1 and v_2 be two strictly increasing, continuous and surjective self-maps on X such that v_1 is strictly convex and v_2 is strictly concave. Now define $\succsim_i$ on $\blacktriangle(X)$ by

$$(\langle \mathbf{p} \rangle, f) \succsim_i (\langle \mathbf{q} \rangle, g) \quad \text{iff} \quad \sum_{x \in \text{supp}(\mathbf{p})} \mathbf{p}(x)v_i^{-1}(\mathbb{E}(v_i, f(x))) \geq \sum_{x \in \text{supp}(\mathbf{q})} \mathbf{q}(x)v_i^{-1}(\mathbb{E}(v_i, g(x)))$$

for $i = 1, 2$. It is easily verified that $\succsim$ is uncertainty neutral, $\succsim_1$ is (strictly) uncertainty averse and $\succsim_2$ is (strictly) uncertainty loving, but we have $\Phi(\succsim_1) = \Phi(\succsim) = \Phi(\succsim_2)$. If we were to adopt the intuitive "definition" we gave above, we would have to conclude that the risk preferences $\Phi(\succsim)$ are (strictly) pessimistic as well as (strictly) optimistic, which is not sensible.[11] $\square$

This discussion demonstrates that not much can be said about the pessimism/optimism of an individual through subjective risk theory unless the analyst is willing to take a stand on what kind of procedure the decision maker uses in her mind in evaluating info-acts and to impose some structure on her preferences. Fortunately, there is a useful way of doing this.

Viable Classes of Preferences. It is clear at this point that to have any chance of obtaining a proper definition of pessimism via the mapping Φ as we have outlined above, the domain of Φ must be restricted so that this map acts in a one-to-one manner on that restricted domain. This prompts the following:

Definition. A subset $\mathcal{S}$ of $\mathfrak{P}(X)$ is said to be **viable** if the equality of any two preference relations in $\mathcal{S}$ over the set of all degenerate info-acts implies the equality of these relations on the entire $\blacktriangle(X)$, that is, $\succsim_1 |\blacktriangle_{\text{d}}(X) = \succsim_2 |\blacktriangle_{\text{d}}(X)$ implies $\succsim_1 = \succsim_2$ for every $\succsim_1$ and $\succsim_2$ in $\mathcal{S}$.[12]

It follows easily from the definition of the map Φ that a subset $\mathcal{S}$ of $\mathfrak{P}(X)$ is viable if and only if $\Phi|_{\mathcal{S}}$ is injective. This observation makes the viable classes of preferences essential for the present exercise. If the analyst is willing to assume that the preferences of an individual over info-acts come from a given viable class, then every risk preference that this individual may have has a unique preimage in $\mathfrak{P}(X)$, and hence, we may consistently detect the pessimism/optimism of the former through the uncertainty aversion/lovingness of the latter.

[11] Another way of looking at the source of this difficulty is the following: The preference relation $\Phi(\succsim)$ admits multiple general weighted EU representations, one in which the probabilities are distorted towards worse outcomes, one in which this happens towards better outcomes, and one in which probabilities are unaltered. (This observation, and indeed the gist of Example 1 is due to Dillenberger, Postlewaite and Rozen (2012).) It is simply impossible to classify such risk preferences on the pessimism-optimism scale unless we presume more information about the structure of these preferences.

[12] Every singleton subset of $\mathfrak{P}(X)$ is viable. Consequently, Example 1 shows that the union of two viable classes need not be viable. On the other hand, the viability property is closed under intersections. It is also hereditary in the sense that every subset of a viable class is viable.

Pessimism with respect to a Viable Class. Given a viable class of preferences, the intuitive definition of pessimism we have given at the start of this section is formalized as follows:

Definition. Let $\mathcal{S}$ be a viable subset of $\mathfrak{P}(X)$. We say that a preference relation $\trianglerighteq$ on $\triangle(X)$ is **pessimistic with respect to** $\mathcal{S}$, or more succinctly, that it is $\mathcal{S}$**-pessimistic**, if there is an uncertainty averse preference relation $\succsim$ in $\mathcal{S}$ such that $\trianglerighteq = \Phi(\succsim)$.

This notion is well-defined in the sense that $\trianglerighteq = \Phi(\succsim_1)$ and $\trianglerighteq = \Phi(\succsim_2)$ cannot hold for two preference relations $\succsim_1$ and $\succsim_2$ in $\mathcal{S}$, one uncertainty averse and the other not. After all, due to the viability of $\mathcal{S}$, there can be at most one $\succsim$ in $\mathcal{S}$ with $\trianglerighteq = \Phi(\succsim)$. However, a preference relation $\trianglerighteq$ in $\mathfrak{R}(X)$ may well be pessimistic with respect to some viable $\mathcal{S}$, but not so with respect to another viable subset $\mathcal{T}$ of $\mathfrak{P}(X)$. (Consider Example 1, for instance.) This is hardly exceptionable. The subjective risk model stipulates a particular mind set for the evaluation of info-acts, and reads off one's risk preferences from those evaluations. "Pessimism" in the risk world is detected as a special case of a form of preference for hedging (that is, uncertainty aversion) that is prevalent in a larger domain (namely, in that of preferences over info-acts). As such, it can be identified only relative to the "model of the mind" of the agent, that is, relative to the viable class under consideration.[13]

The definition of viability given above is a rather indirect one. Loosely speaking, this definition makes it clear what a viable class is good for, but it gives hardly any clue as to what such classes may look like. Indeed, if a viable class is necessarily small, or uninteresting as a model of preferences over info-acts, then defining pessimism for risk preferences as above would not be of much use. Fortunately, there are fairly large viable classes, and some of these stem from models commonly used in the theory of choice under uncertainty. We will introduce here two such examples, and in the next section use these examples to classify certain types of risk preferences as pessimistic or not.

Example: The Class of Biseparable Preferences. Let Σ be a finite algebra on Ω and $\mathcal{F}$ the set of all Σ-measurable maps from Ω into $\triangle(X)$ with finite range. A continuous map $V : \mathcal{F} \to \mathbb{R}$ is called **biseparable** if there exists a set-function $\rho : \Sigma \to [0, 1]$ such that $\rho(\emptyset) = 0$, $\rho(\Omega) = 1$, and

$$V(\delta_x \mathbf{1}_S + \delta_y \mathbf{1}_{\Omega \backslash S}) = \rho(S)V(\delta_x \mathbf{1}_\Omega) + (1 - \rho(S))V(\delta_y \mathbf{1}_\Omega) \tag{8}$$

for every $S \in \Sigma$ and $x, y \in X$ with $x \geq y$. If, in addition, this map evaluates constant acts by their expected utility, that is,

$$V(\mathbf{p}\mathbf{1}_\Omega) = \sum_{x \in \text{supp}(\mathbf{p})} V(\delta_x \mathbf{1}_\Omega)\mathbf{p}(x)$$

for every $\mathbf{p} \in \triangle(X)$, we say that V is a **biseparable EU function**.[14]

[13]There is an obvious caveat here. If one chooses a small viable class, and considers pessimism only with respect to this class, this concept, while well-defined, would be of limited use, both conceptually and from the perspective of applications. (After all, and we reiterate, a viable class in the present setup stands for a model of one's unobservable preferences over info-acts, so the more restrictive assumptions one imposes on that model, the less appealing will be one's related findings.)

[14]Ghirardato and Marinacci (2001) refers to a preference relation on $\Delta(X)$ that is represented by such a map V as a *c-linear biseparable preference relation.*

Recall that a preference relation $\succsim$ on $\mathcal{F}$ is said to be **monotonic** if $f \succsim g$ holds for every $f, g \in \mathcal{F}$ such that $f(\omega)\mathbf{1}_\Omega \succsim g(\omega)\mathbf{1}_\Omega$ for each $\omega \in \Omega$. Combining these two notions, we say that a preference relation $\succsim$ on $\mathcal{F}$ has a **biseparable EU representation** (or that it is a **biseparable EU preference**) if $\succsim$ is monotonic, it deems more money preferable to less (that is, $\delta_x\mathbf{1}_\Omega \succsim \delta_y\mathbf{1}_\Omega$ iff $x \geq y$), and it can be represented by a biseparable EU function.

The class of preferences with biseparable EU representations were introduced by Ghirardato and Marinacci (2001, 2002). Roughly speaking, this class consists of those monotone preferences that evaluate those acts that yield the same (objective) lottery at all states, and those comonotonic acts with only two different (certain) outcomes according to the Anscombe-Aumann expected utility theory. This class is quite rich. It contains a variety of preferences, including preferences that correspond to the subjective expected utility (SEU), Choquet expected utility (CEU), maxmin expected utility (MMEU), as well as preferences of Hurwicz type. Furthermore, it is straightforward to extend this notion to the case of preferences over info-acts: A preference relation $\succsim$ in $\mathfrak{P}(X)$ is said to be a **biseparable EU preference** if $\succsim^{\mathbf{p}}$ has a biseparable EU representation for every $\mathbf{p} \in \triangle(X)$. The importance of such preferences for the present exercise stems from the following fact:

Proposition 3. *Any collection of biseparable EU preference relations on $\blacktriangle(X)$ is viable.*[15]

In other words, the map Φ from $\mathfrak{P}(X)$ into $\mathfrak{R}(X)$ is injective on the class of all biseparable EU preferences on $\blacktriangle(X)$. Therefore, this map is left-invertible on this class, that is, provided that one's preferences over info-acts are of biseparable EU type, we can identify these preferences by observing the choices of this individual over monetary lotteries. We will demonstrate in the next section that a good deal follows from this observation.[16]

Example: The Class of c-Neutral Preferences. For any preference relation $\succsim$ in $\mathfrak{P}(X)$, we say that $\succsim$ is a **c-neutral preference** if $\succsim^{\mathbf{p}}$ is (i) monotonic, and (ii) evaluates constant acts in a risk neutral manner for any $\mathbf{p}$ in $\triangle(X)$, that is,

$$\mathbf{q}\mathbf{1}_\Omega \sim^{\mathbf{p}} \delta_{\mathbb{E}(\mathbf{q})}\mathbf{1}_\Omega \quad \text{for every } \mathbf{q} \in \triangle(X) \tag{9}$$

for each simple lottery $\mathbf{p}$ on X. Put differently, a c-neutral preference relation on $\blacktriangle(X)$ is monotonic and declares any given info-act that pays off the same $\mathbf{q}$ in every state to the info-act that pays the expected value of $\mathbf{q}$ in every state (with the same info). We will find such preferences particularly useful when examining the connection between the notions of pessimism and risk aversion. For now we note that such preferences too form a viable class.

Proposition 4. *Any collection of c-neutral preference relations on $\blacktriangle(X)$ is viable.*

[15] There are larger viable subclasses of $\mathfrak{P}(X)$ that contain all biseparable EU preferences. For instance, Ghirardato and Pennesi (2012) introduce a generalization of biseperable EU preferences where the map ρ in (8) is allowed to depend on x and y. The collection of all such preferences too is a viable class, but we will not need this fact in this paper.

[16] While Proposition 3 shows that there are large viable classes of preferences, we should note that there are also interesting classes of preferences on $\blacktriangle(X)$ that are not viable. For instance, consider the class of all preferences $\succsim$ on $\blacktriangle(X)$ such that $\succsim^{\mathbf{p}}$ is a multiplier preference for every $\mathbf{p} \in \Delta(X)$ (cf. Hansen and Sargent (2001)). It is not difficult to show that this class is not viable.

Clearly, a c-neutral preference relation need not be of the biseparable EU form, nor is a biseparable EU preference necessarily c-neutral. As such, Propositions 3 and 4 are logically distinct. However, unlike the former, Proposition 4 is fairly straightforward. Indeed, if $\succsim$ is a c-neutral preference in $\mathfrak{P}(X)$, then $f(x)\mathbf{1}_\Omega \sim^{\mathbf{p}} \delta_{\mathbb{E}(f(x))}\mathbf{1}_\Omega$, and thus, by monotonicity,

$$f \sim^{\mathbf{p}} \sum_{x\in\mathrm{supp}(\mathbf{p})} \delta_{\mathbb{E}(f(x))}\mathbf{1}_{\Omega(\mathbf{p},x)}.$$

for every $(\langle\mathbf{p}\rangle, f) \in \blacktriangle(X)$. This means that such a preference relation is determined entirely on the basis of its behavior on $\blacktriangle_{\mathrm{d}}(X)$, and hence Proposition 4.

4.2 Applications

This section is devoted to the analysis of three concrete classes of risk preferences. Our objective is to characterize when a preference relation that belongs to any one of these classes would be viewed as pessimistic (or optimistic) with respect to a particular viable class.

4.2.1 The Rank-Dependent Utility Model

The RDU Model. The most widely applied case of the general weighted EU model is the *rank-dependent utility* (RDU) *theory*. According to this theory, one's risk preferences $\trianglerighteq$ in $\mathfrak{R}(X)$ are represented by a utility function $U : \triangle(X) \to \mathbb{R}$ of the form

$$U(\mathbf{p}) := \sum_{k=1}^{n-1} \left(w(p_k + \cdots + p_n) - w(p_{k+1} + \cdots + p_n)\right) u(x_k) + w(p_n)u(x_n) \tag{10}$$

where $\{x_1, ..., x_n\}$ is the support of $\mathbf{p}$ with $x_1 < \cdots < x_n$, $u : X \to \mathbb{R}$ is a continuous and strictly increasing function, and w is a continuous and strictly increasing self-map on $[0, 1]$ such that $w(0) = 0$ and $w(1) = 1$. (In this case we refer to $\trianglerighteq$ as an **RDU preference on** $\triangle(X)$ **which is represented by** (w, u).) In terms of the general weighted EU model (4), we see here that the objective probability $\mathbf{p}(x)$ of an outcome x in $\mathrm{supp}(\mathbf{p})$ is distorted to

$$\pi(x, \mathbf{p}) = w(\mathbf{p}(x) + \mathbf{p}\text{-rank of } x) - w(\mathbf{p}\text{-rank of } x), \tag{11}$$

where the **p-rank** of x is defined as the probability of receiving an outcome that is strictly better than x in the lottery $\mathbf{p}$, that is,

$$\text{the } \mathbf{p}\text{-rank of } x := \sum_{\substack{\omega\in\mathrm{supp}(\mathbf{p})\\ \omega>x}} \mathbf{p}(\omega).$$

Pessimism in the RDU Model. It follows easily from (11) that $\pi(x, \cdot)$ is increasing in the $\mathbf{p}$-rank of x[17] iff w is a convex function. Put differently, when w is convex, the probabilities of a given lottery are sure to be distorted so that the outcomes with higher ranks in that lottery are given higher probability weights. It is thus quite intuitive that an RDU preference on $\triangle(X)$

[17] By this, we mean that, for any two simple lotteries $\mathbf{p}$ and $\mathbf{q}$ on X whose supports include x, we have $\pi(x, \mathbf{p}) \geq \pi(x, \mathbf{q})$ whenever the $\mathbf{p}$-rank of x is as large as the $\mathbf{q}$-rank of x and $\mathbf{p}(x) = \mathbf{q}(x)$.

which is represented by (w, u) would be deemed "pessimistic" if w is convex, and indeed, this is precisely how pessimism is defined in the literature within the RDU model.

The RDU model provides a nice check for the appeal of our definition of pessimism. Indeed, within this domain, it seems quite clear how pessimism should be captured, and a definition that strays away from this would be suspect. Fortunately, in the context of the entire class of biseparable EU preferences on $\blacktriangle(X)$ – recall Proposition 3 – our definition of pessimism matches that defined for RDU preferences exactly.

Proposition 5. *Let $\mathcal{S}$ stand for the class of all biseparable EU preference relations on $\blacktriangle(X)$, and let $\trianglerighteq$ be an RDU preference relation on $\triangle(X)$ which is represented by (w, u). Then, $\trianglerighteq = \Phi(\succsim)$ for some $\succsim$ in $\mathcal{S}$. Furthermore, $\trianglerighteq$ is $\mathcal{S}$-pessimistic if, and only if, w is a convex function.*

We are thus able to conform with the intuition derived within the RDU framework about pessimism through the foundations provided by preferences over info-acts (so long as those preferences are of the biseparable EU form).

4.2.2 The Multi-RDU Model

In a recent paper, Dean and Ortoleva (2013) have investigated risk preferences $\trianglerighteq$ in $\mathfrak{R}(X)$ that are represented by a utility function $U : \triangle(X) \to \mathbb{R}$ of the form

$$U(\mathbf{p}) := \min_{w \in \mathcal{W}} \left(\sum_{k=1}^{n-1} \left(w(p_k + \cdots + p_n) - w(p_{k+1} + \cdots + p_n) \right) u(x_k) + w(p_n) u(x_n) \right) \tag{12}$$

where $\{x_1, ..., x_n\}$ is the support of $\mathbf{p}$ with $x_1 < \cdots < x_n$, $u : X \to \mathbb{R}$ is a continuous and strictly increasing function, and $\mathcal{W}$ is a nonempty convex set of continuous, strictly increasing and convex self-maps on $[0, 1]$ that admit 0 and 1 as fixed points. A fairly straightforward consequence of Proposition 5 is that any such a preference relation is pessimistic with respect to the viable class $\mathcal{S}$ of biseparable EU preference relations on $\blacktriangle(X)$.

There is, however, a more intimate connection between the work of Dean and Ortoleva (2013) and our approach toward modeling pessimism. That paper provides an axiomatic characterization of the above class of preferences, and one of its key axioms, called "Hedging axiom," is primed to capture the potential pessimism of a decision-maker. This property utilizes outcome-mixtures in the style of Ghirardato et al. (2003) to formulate the notion of hedging in the context of lotteries, and roughly speaking, says that a pessimistic person is one that exhibits preferences for hedging because this reduces the variance of utility-outcomes. While it is formally distinct from the way we have defined pessimism above, it is clear that the motivation behind the Dean-Ortoleva formulation of pessimism is similar to that behind our approach. In fact, it turns out that there is a formal connection between the two formulations as well. Let $\trianglerighteq$ be a preference relation in $\mathcal{R}(X)$ such that $\trianglerighteq = \Phi(\succsim)$ for some $\succsim$ in $\mathcal{S}$. Then, one can show that $\trianglerighteq$ is $\mathcal{S}$-pessimistic iff it satisfies the Hedging axiom.[18] Thus, the

[18] As this is mostly a side remark, we omit the proof of this fact here. It is, however, available from the authors upon request.

two approaches to modeling pessimism are in sync, at least in the context of biseparable EU preference relations.[19]

4.2.3 Disappointment Aversion and Pessimism

Gul's Model. We consider next another important special case of the general weighted EU model, namely, Gul's 1991 model of risk preferences that allows for disappointment aversion or elation loving. This model is described as follows. Take a risk preference $\trianglerighteq$ in $\mathfrak{R}(X)$, and for any $\mathbf{p} \in \triangle(X)$, define

$$B(\mathbf{p}, \trianglerighteq) = \{\mathbf{q} \in \triangle(X) : \delta_x \trianglerighteq \mathbf{p} \text{ for all } x \in \text{ supp}(\mathbf{q})\}$$

and

$$W(\mathbf{p}, \trianglerighteq) = \{\mathbf{r} \in \triangle(X) : \mathbf{p} \trianglerighteq \delta_x \text{ for all } x \in \text{ supp}(\mathbf{r})\}.$$

A triplet $(\alpha, \mathbf{q}, \mathbf{r})$ is said to be an **elation/disappointment decomposition** of $\mathbf{p} \in \triangle(X)$ **with respect to** $\trianglerighteq$ if $0 \leq \alpha \leq 1$, $\mathbf{q} \in B(\mathbf{p}, \trianglerighteq)$, $\mathbf{r} \in W(\mathbf{p}, \trianglerighteq)$ and $\mathbf{p} = \alpha\mathbf{q} + (1-\alpha)\mathbf{r}$. (If the certainty equivalent of $\mathbf{p}$ belongs to its support, there would be (infinitely) many such decompositions.)

We say that a $\trianglerighteq \in \mathfrak{R}(X)$ is a **Gul preference** if it can be represented by a utility function $U : \triangle(X) \to \mathbb{R}$ of the form

$$U(\mathbf{p}) = \gamma(\alpha) \sum_{x \in \text{supp}(\mathbf{q})} \mathbf{q}(x)u(x) + (1-\gamma(\alpha)) \sum_{x \in \text{supp}(\mathbf{r})} \mathbf{r}(x)u(x), \tag{13}$$

where $(\alpha, \mathbf{q}, \mathbf{r})$ is an elation/disappointment decomposition of $\mathbf{p}$ with respect to $\trianglerighteq$, $u : X \to \mathbb{R}$ is a continuous and strictly increasing function, and γ is a real map on $[0, 1]$ such that

$$\gamma(t) = \frac{t}{1 + (1-t)\beta}, \quad 0 \leq t \leq 1,$$

for some real number $\beta > -1$. (In this case we refer to $\trianglerighteq$ as a **Gul preference on** $\triangle(X)$ **which is represented by** (β, u).) Such a preference relation is said to be **disappointment averse** if $\beta \geq 0$, and **elation loving** if $-1 < \beta \leq 0$. It collapses to a standard expected utility preference when $\beta = 0$.

Pessimism in Gul's Model. Intuitively speaking, the construction of Gul's preferences is based on decomposing the support of a lottery into "good" outcomes and "bad" outcomes, and assigning different weights into these in the ensuing expected utility computation. Besides, it is clear that the higher β, the lower the γ (everywhere on its domain), and the more weight is assigned to "bad" outcomes in this representation. As $\beta = 0$ corresponds to the case of expected utility preferences, it is thus natural to view the case $\beta > 0$ as corresponding to the

[19] The link between the two approaches are broken when we go outside the biseparable EU preference relations. Put differently, if $\trianglerighteq = \Phi(\succsim)$ does not hold for any $\succsim$ in $\mathcal{S}$, then the said equivalence fails. Perhaps more important is the fact that, without this assumption (which roughly means the failure of the tradeoff-consistency axiom of Dean and Ortoleva (2013)), the Hedging axiom loses much of its appeal (as it becomes incompatible with outcome-mixtures other than the 50-50 ones).

case of pessimism. That is, it seems quite reasonable to call pessimism to what Gul refers to as disappointment aversion, and similarly, identify elation loving with optimism. As it turns out, the definition of pessimism we introduced above (and again with respect to the viable class of biseparable EU preferences) does precisely this.

Proposition 6. *Let $\mathcal{S}$ stand for the class of all biseparable EU preference relations on $\blacktriangle(X)$, and let $\trianglerighteq$ be a Gul preference relation on $\triangle(X)$ which is represented by (β, u). Then, $\trianglerighteq = \Phi(\succsim)$ for some $\succsim$ in $\mathcal{S}$. Furthermore, $\trianglerighteq$ is $\mathcal{S}$-pessimistic if, and only if, $\beta \geq 0$.*

Gul's model thus provides another instance of the general weighted EU theory in which the pessimism notion we advance here turns out to yield the "correct" answer. This notion (with respect to the viable class used above) coincides in this model with disappointment aversion. We will see in Section 4.3 that this would not at all be the case if we tried to define pessimism for this model through probability distortions as in the case of the RDU model.

4.2.4 The Cautious Expected Utility Model

In the previous two examples we have dealt with two models of risk preferences relative to which one has a strong feeling for what "pessimism" means. Our approach toward modeling pessimism thus conforms with intuition in the context of these examples. In our final application, we look at a type of risk preference relative to which it is not intuitively clear how one's pessimism could be captured.

Cautious Expected Utility. The *cautious expected utility theory* is developed in the recent work of Cerreia-Vioglio, Dillenberger and Ortoleva (2013). This theory provides an insightful axiomatic characterization for those preference relations $\trianglerighteq \in \mathfrak{R}(X)$ which can be represented by a utility function $U : \triangle(X) \to \mathbb{R}$ of the form

$$U(\mathbf{p}) := \inf_{v \in \mathcal{V}} v^{-1}(\mathbb{E}(v, \mathbf{p})), \tag{14}$$

where $\mathcal{V}$ is a nonempty set of continuous and strictly increasing real maps on X such that (i) $v(\min X) = 0 = 1 - v(\max X)$ for every $v \in \mathcal{V}$, and (ii) U is continuous. Given such a $\trianglerighteq$ and $\mathcal{V}$, we say that $\trianglerighteq$ is a **cautious EU preference on** $\triangle(X)$ **which is represented by** $\mathcal{V}$, or equivalently, that $\mathcal{V}$ is a **cautious EU representation for** $\trianglerighteq$.

Not every such preference arises through Φ by a biseparable EU preference on info-acts, so examining the pessimism properties of cautious EU preferences requires using a different viable class. It is easily seen that the class of c-neutral preferences can be used for this purpose. Indeed, take any cautious EU preference $\trianglerighteq$ on $\triangle(X)$ which is represented by $\mathcal{V}$, and consider the map $U^* : \blacktriangle(X) \to \mathbb{R}$ defined by

$$U^*(\langle \mathbf{p} \rangle, f) := \inf_{v \in \mathcal{V}} v^{-1} \left(\sum_{x \in \operatorname{supp}(\mathbf{p})} \mathbf{p}(x) v(\mathbb{E}(f(x))) \right). \tag{15}$$

It is easy to see that $\trianglerighteq = \Phi(\succsim)$ where $\succsim$ is the preference relation on $\blacktriangle(X)$ which is represented by U^*. Besides, it is also routine to verify that $\succsim$ belongs to $\mathfrak{P}(X)$ and is c-neutral. We may thus conclude that every cautious EU preference relation in $\mathfrak{R}(X)$ is the image of some c-neutral preference relation in $\mathfrak{P}(X)$ under Φ. As the collection of all c-neutral preferences in $\mathfrak{P}(X)$ is viable (Proposition 4), it is meaningful to classify cautious EU preference on the basis of pessimism relative to the class of c-neutral preferences on $\blacktriangle(X)$. This leads us to an interesting observation:

Proposition 7. *Let $\mathcal{S}$ stand for the class of all c-neutral preference relations on $\blacktriangle(X)$, and let $\trianglerighteq$ be a cautious EU preference on $\triangle(X)$ which is represented by V. Then, $\trianglerighteq = \Phi(\succsim)$ for some $\succsim$ in $\mathcal{S}$. Furthermore, $\trianglerighteq$ is $\mathcal{S}$-pessimistic if, and only if, it is risk averse.*

In the context of the cautious EU model, we thus find that the notion of pessimism (with respect to the class of c-neutral preference relations) reduces to the standard property of risk aversion.

Remark. In passing, we note that it is also possible to characterize the pessimism of $\trianglerighteq$ functionally in the context of Proposition 7. Indeed, if every v in $\mathcal{V}$ is concave, then $\trianglerighteq$ is risk averse, and hence, $\mathcal{S}$-pessimistic. Conversely, if $\trianglerighteq$ is $\mathcal{S}$-pessimistic, then there is a subset $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{W}$ is a cautious EU representation for $\trianglerighteq$ and every element of $\mathcal{W}$ is concave.[20]

4.3 The General Weighted EU Model and Pessimism

Intuitively, an individual is considered "pessimistic" in the literature if she distorts the probability of outcomes in such a way that she views low outcomes as more likely than they actually are. As such, this notion is defined in the literature only for some special cases of the general weighted EU model. To be precise, let us consider a preference relation $\trianglerighteq$ in $\mathfrak{R}(X)$ that can be represented by a utility function $U : \triangle(X) \to \mathbb{R}$ of the form (4) where $u : X \to \mathbb{R}$ is a continuous and strictly increasing function and π a probability weighting function on X. Given that we interpret $\pi(x, \mathbf{p})$ as the distorted probability of receiving x in the lottery, it seems reasonable to define "pessimism" by suitably comparing the distorted probabilities with the original ones (in the case of all lotteries).

Recall that, for any $\mathbf{p} \in \triangle(X)$ and $x \in \mathrm{supp}(\mathbf{p})$, the $\mathbf{p}$-*rank* of x is the probability of receiving an outcome that is strictly better than x in the lottery $\mathbf{p}$. This concept allows us to understand how good x is relative to the other outcomes in the support of $\mathbf{p}$. The higher the $\mathbf{p}$-rank order of x, the worse this outcome is relative to these other outcomes. This prompts the following formulation which is commonly adopted in the literature:

Definition. Let $\trianglerighteq$ be a general weighted EU preference on $\triangle(X)$ which is represented by (π, u). We say that $\trianglerighteq$ is **π-pessimistic** if for any $\mathbf{p}$ and $\mathbf{q}$ in $\triangle(X)$, and any x that belongs to the supports of both $\mathbf{p}$ and $\mathbf{q}$, we have $\pi(x, \mathbf{p}) \geq \pi(x, \mathbf{q})$ whenever the $\mathbf{p}$-rank of x is as large as the $\mathbf{q}$-rank of x and $\mathbf{p}(x) = \mathbf{q}(x)$. (We define **$\pi$-optimism** dually.)

Unfortunately, there are some difficulties with this definition. First of all, defining pessimism through the probability weighting function used in a representation of the form (4)

[20] The latter statement obtains by combining Theorem 3 of Cerreia-Vioglio, Dillenberger and Ortoleva (2013) with our Proposition 7. Loosely speaking, the reason why we need to consider a subset of $\mathcal{V}$ instead of the entire $\mathcal{V}$ is because this set may include functions that are redundant for the computation of the utility function given in (14).

is, in general, behaviorally untenable. This is because π and u are not uniquely determined in the representation (4). It is not difficult to find two pairs (π, u) and (π', u') that represent the same preference relation on $\triangle(X)$ such that this preference relation is π-pessimistic but not π'-pessimistic. Put differently, in the general weighted EU model, the definition of π-pessimism does not depend only on $\trianglerighteq$ for which we have a representation by (π, u).

This difficulty is well known. In response, most authors adopt the definition of pessimism above in certain special cases of the general weighted EU model in which π is uniquely identified (and u is determined up to positive affine transformations). Obviously, this is not unrelated to why we had to introduce the notion of a viable class above. However, the nature of the dependence of our formulation of pessimism on the choice of the viable class is different than how "π-pessimism" concept depends on the class of general weighted EU preferences. First, the former definition is behaviorally meaningful, while, as we have noted above, the latter is not so in general. Second, a viable class is not necessarily characterized by a particular functional form, but rather by means of either structural or behavioral properties. For instance, we have seen earlier that the collection of all biseparable EU preferences, as well as that of all c-neutral preferences, on $\blacktriangle(X)$ are viable, but these classes are not characterized by functional form specifications. (The former is determined by structural properties, while the latter purely behavioral properties.) Finally, and more importantly, by choosing different viable classes, one can trace entirely different types of risk preferences (by looking at the image of these classes under Φ). By varying the viable classes, therefore, the present definition of pessimism modifies and becomes applicable to essentially any type of risk preferences.

Pessimism in the RDU Model, Revisited. In the representation (10) w is uniquely identified (and u is unique up to positive affine transformations). Put differently, within the RDU theory, the probability weighting function π is uniquely defined by one's preferences, and thus the definition of π-pessimism is behaviorally meaningful. Furthermore, in that model, this definition leads to a nice characterization: An individual whose preference relation on $\Delta(X)$ is an RDU preference which is represented by (w, u) is π-pessimistic (where π is defined by (11)) iff w is a convex function. (π-Optimism is captured dually.)[21] In view of Proposition 5, therefore, we find that the notion of pessimism adopted in the literature for the RDU model is in full accord with the way we have defined this concept (with respect to the viable class of biseparable preferences).

A natural query is if this observation extends to other subclasses of the general weighted EU theory (in which the probability weighting functions are uniquely identified). Put differently, is it the case that the notion of pessimism we have introduced via the subjective risk model is none other than how pessimism is defined in the context of general weighted EU theory (whenever the latter conceptualization of pessimism in terms of probability distortions is behaviorally meaningful)? The answer is no, as we show next.

Pessimism in Gul's Model, Revisited. Let $\trianglerighteq$ be a Gul preference relation on $\triangle(X)$ which is represented by (β, u). It can be shown that $\trianglerighteq$ is of the general weighted EU form, that is,

[21] This observation is stressed, for instance, in Wakker (1994) and Abdellaoui (2002), among numerous other authors. It is common in the literature on RDU theory to take the convexity of w as the "definition" of pessimism.

it can be represented by a utility function of the form (4) for some probability weighting function π and u. Unlike the RDU model, however, the utility function U that represents $\trianglerighteq$ as in Gul's model is only implicitly defined, so we cannot in general provide a closed-form description of the map π in terms of β and u. However, as proved in Theorem 1 of Gul (1991), β is unique here and u is unique up to positive affine transformations, so π is in fact uniquely identified. Consequently, as in the RDU model, it is behaviorally meaningful to view $\trianglerighteq$ as pessimistic when it is π-pessimistic. Unfortunately, in this case, this way of looking at the trait of pessimism does not turn out to be useful:

Proposition 8. *Let $\trianglerighteq$ be a Gul preference relation on $\triangle(X)$, and let this relation be represented by (π, u) as in the general weighted EU model. Then, the following are equivalent*: (i) *$\trianglerighteq$ is π-pessimistic*; (ii) *$\trianglerighteq$ is an expected utility preference*; (iii) *$\trianglerighteq$ is π-optimistic.*

In words, an attempt to extend the rank-based definition of "pessimism" to Gul's model cannot classify any preferences in that model on the pessimism-optimism scale, with the exception of the standard expected utility preferences (which are neutral so far as that scale is concerned). Looking at pessimism by means of probability weighting functions is simply too coarse to declare an agent with Gul preferences as pessimistic in a nontrivial manner. Gul's model thus provides an instance of the general weighted EU theory in which the definition of pessimism that is commonly adopted in the literature does not deliver an intuitive answer, but the pessimism notion we advance here turns out to yield arguably the "correct" answer. As we have found in Proposition 6, the latter notion (with respect to the viable class of biseparable EU preferences) coincides in this model with disappointment aversion.

Pessimism in the Cautious Expected Utility Model, Revisited. Let $\trianglerighteq$ be a cautious EU preference on $\triangle(X)$ which is represented by $\mathcal{V}$. This preference too is of the general weighted EU form, so we may think of classifying it as pessimistic or not by using the π-pessimism notion. However, in this case, this notion is not even operational. For, as we have noted above, this definition is meaningful only on subclasses of general weighted EU preferences in which the probability weighting function is unique and the utility function is unique up to positive affine transformations. So, to use the π-pessimism concept for the cautious expected utility model, we need to see this model as a subcollection of all general weighted EU preferences which is represented by (π, u) such that (π, u) satisfies some conditions (that ensure their uniqueness). This is exactly what we have done above with the RDU model and Gul's model, but there was a natural way of choosing the (π, u)s for the representations in those models, and hence, it was clear which subcollections of the general weighted EU preferences are to be used. Insofar as we can see, this is not the case for the cautious EU model. There does not seem to be a natural way of viewing $\trianglerighteq$ as represented in the general weighted EU form by means of a particular pair (π, u) (and of course, there are infinitely many such pairs that are up to the task). Put differently, the standard practice of limiting attention to a particular subclass of general weighted EU preferences and then applying the π-pessimism notion does not work (at least in a straightforward manner) in the context of the cautious EU theory. As we have seen in Proposition 7, however, our formulation of pessimism (with respect to the viable class of c-neutral preferences) does not encounter such a difficulty, and classifies a cautious EU preference as pessimistic iff it is risk averse.

5 Pessimism and Probabilistic Risk Aversion

Probabilistic Risk Aversion. In contrast to the classical expected utility theory under risk, the characterization of risk aversion is often a complicated matter in the context of non-expected utility theories. For instance, in the case of a preference relation on $\Delta(X)$ that is represented by a utility function of the form (4) as in the generalized weighted EU model, attitudes toward risk arise both from the structure of the probability weighting function π (distortion of probabilities) and the curvature of u (utility for money). The former effect, that is, the influence of probability weighting on risk attitudes is referred to as *probabilistic risk attitude* in the literature. To understand this effect in isolation, one may take u as an affine function in (4), and then characterize the risk attitudes of preferences in terms of the structure of the probability weighting function (provided that this function is uniquely determined). For instance, one can look for a functional characterization of the probability weighting functions that represent risk averse preferences when coupled with an affine utility function for money. Informally speaking, such a characterization would tell us when the agent can be viewed as *probabilistically risk averse*, that is, when her probability distortions contribute positively to her potential risk aversion.[22]

In the subjective risk model, we obtain the distortions of objectively given probabilities from one's preferences over info-acts. Consequently, we may ask what sort of behavioral traits of the latter type of preferences would be responsible for inducing distortions of probabilities that exhibit particular types of probabilistic risk attitudes. We shall investigate this query here in terms of probabilistic risk aversion. This investigation parallels the one we have reported in Section 3. We have seen there that deviations from ambiguity neutrality (in terms of preferences over info-acts) is a necessary condition for the resulting risk preferences to diverge from the expected utility model (and hence some distortion of objective probabilities to occur). We shall now attempt to identify what must be true for one's preferences over info-acts so that the induced risk preferences exhibit probabilistic risk aversion. Put succinctly, we wish to utilize the subjective risk theory to find a "source" for one's probabilistic risk aversion.

Uncertainty Aversion implies Probabilistic Risk Aversion. To make things precise, let us recall that a simple lottery $\mathbf{q}$ on X is a **mean-preserving spread** of another such lottery $\mathbf{p}$ if, where $\text{supp}(\mathbf{p})$ is enumerated as $\{x_1, ..., x_k\}$, there exist simple lotteries $\theta_1, ..., \theta_k$ on X such that

$$\mathbf{q} = \sum_{i=1}^{k} \mathbf{p}(x_i)\theta_i \quad \text{and} \quad \mathbb{E}(\theta_i) = x_i \text{ for each } i = 1, ..., k.$$

In turn, a preference relation $\trianglerighteq$ on $\Delta(X)$ is said to be **risk averse** if $\mathbf{p} \trianglerighteq \mathbf{q}$ for every $\mathbf{p}, \mathbf{q} \in \Delta(X)$ such that $\mathbf{q}$ is a mean-preserving spread of $\mathbf{p}$.[23]

Our objective here is to understand what sort of properties of a preference relation $\succsim$ on $\blacktriangle(X)$ entails the risk aversion of $\Phi(\succsim)$ insofar as this arises due to one's subjective assessment of the likelihoods in a given lottery. Obviously, this issue is irrelevant for the comparative

[22] We stress that a probabilistically risk averse agent need not be risk averse in the final analysis. The overall attitude toward risk requires the joint input of both the probability weighting function and the utility function.

[23] In the context of non-expected utility theory, what we define here as risk aversion is often referred to as *strong* risk aversion.

evaluation of two *constant* info-acts (whose "info" parts are the same). Consequently, we shall impose that the preferences at hand act in a risk neutral manner when comparing such info-acts, that is, we shall impose (9) on $\succsim$. If $\succsim$ satisfies this property, any risk aversion that may be detected in the induced risk preferences $\Phi(\succsim)$ must arise from the way this person evaluates the uncertainty she perceives in a given lottery in terms of the likelihoods of prizes. Therefore, our problem may be stated as follows: For any c-neutral preference relation $\succsim$ on $\blacktriangle(X)$, when would $\Phi(\succsim)$ be a risk averse preference relation on $\Delta(X)$? We now show that a sufficient condition for this is none other than *uncertainty aversion*.

Theorem 9. *Let $\succsim$ be a c-neutral preference relation on $\blacktriangle(X)$. If $\succsim$ is uncertainty averse, then $\Phi(\succsim)$ is risk averse.*

The subjective risk theory thus brings together two empirically meaningful phenomena, uncertainty aversion and risk aversion, that exist in formally disparate realms. Of course, we can restate this connection by using only one's risk preferences as a primitive:

Corollary 10. *Let $\mathcal{S}$ be the class of all c-neutral preference relations on $\blacktriangle(X)$. Then, every $\mathcal{S}$-pessimistic preference relation $\trianglerighteq$ on $\Delta(X)$ is risk averse.*

Informally speaking, therefore, we find that the general notion of pessimism we have introduced in this paper entails the probabilistic risk aversion of one's risk preferences, at least when pessimism is understood relative to the viable class of c-neutral preference relations.

An Open Problem. We have seen in Section 4.2.1 that an RDU preference $\trianglerighteq$ on $\Delta(X)$ which is represented by (w, u) is pessimistic with respect to the biseparable EU class $\mathcal{S}$ iff w is convex. By Propositions 4 and 5, there is a unique biseparable EU preference $\succsim$ on $\blacktriangle(X)$ with $\trianglerighteq = \Phi(\succsim)$. Besides, it is readily checked that u is affine iff $\succsim$ satisfies (9). Consequently, in the context of c-neutral and biseparable EU class, we find that $\trianglerighteq$ is uncertainty averse iff w is convex. But it is well-known that $\trianglerighteq$ is risk averse iff w is convex and u is concave (cf. Chew, Karni and Safra (1987)). It follows that the converse of Theorem 9 is valid, provided that $\Phi(\succsim)$ has an RDU representation. In fact, precisely the same is true when $\Phi(\succsim)$ is a Gul preference, and as we have already noted in Proposition 7, for the cautious EU model. These observations lead us to conjecture that the converse of Theorem 9 is true: *A c-neutral preference relation on $\blacktriangle(X)$ is uncertainty averse if, and only if, the risk preference that it induces is risk averse.* If true, this statement would mean that one's probabilistic risk aversion is but only a reflection of her uncertainty aversion. Finding whether or not the "if" part of this statement is true, however, is left as an open problem at present.

6 Pessimism and Preference for Diversification

Intuitively, we would expect a pessimistic individual to very much dislike "putting all eggs in one basket." Instead, it seems reasonable that she would try to reduce the risk involved in her investments as much as possible by mixing her assets, thereby diversifying her portfolio, *even if her utility for money is a linear function.* The definition of pessimism that is used in non-expected utility theory would be consistent with this intuition only in special cases. That is, the π-pessimism of a general weighted EU preference on $\triangle(X)$ which is represented by (π, u), where u is the identity function, may or may not result in a preference for diversification. In particular, this happens in the context of the RDU model but not in Gul's model (Proposition 8). By contrast, we show in this section that the notion of pessimism we have introduced here (with respect to c-neutral preferences over info-acts) is consistent with this intuition in full generality.

Preferences over Assets. We adopt the model of Dekel (1989) to formalize the problem. Consider the standard probability space $([0,1], \mathcal{B}, \ell)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ and ℓ is the Lebesgue measure. For any random variable x from this space into X, we denote the simple probability measure on X induced by x by $\mathbf{p}_x$, that is, $\mathbf{p}_x := \ell \circ x^{-1}$. In what follows, by an **asset** on X, we mean such a random variable with a finite range. Obviously, for any asset x on X, $\mathbf{p}_x$ is a simple lottery on X. Consequently, the preferences of a decision maker over assets on X are naturally induced by her preference relation $\trianglerighteq$ on $\triangle(X)$. It is convenient to abuse notation and denote the former preferences also by $\trianglerighteq$ here. That is, for any two assets x and y on X, by $x \trianglerighteq y$ we mean simply that $\mathbf{p}_x \trianglerighteq \mathbf{p}_y$.

Pessimism implies Preference for Portfolio Diversification. We say that a preference relation $\trianglerighteq$ over assets on X **exhibits preference for portfolio diversification** if, for any positive integer n and assets $x_1, ..., x_n$ on X,

$$x_1 \simeq \cdots \simeq x_n \quad \text{implies} \quad \lambda_1 x_1 + \cdots + \lambda_n x_n \trianglerighteq x_1$$

for every $\lambda_1, ..., \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$.[24] Our final result in this paper is that if a preference relation over assets on X arises from risk preferences that are pessimistic with respect to the viable class of c-neutral preferences, then it must exhibit preference for portfolio diversification.

Theorem 11. *Let $\mathcal{S}$ be the class of all c-neutral preference relations on $\blacktriangle(X)$. Then, every $\mathcal{S}$-pessimistic preference relation $\trianglerighteq$ on $\Delta(X)$ exhibits preference for portfolio diversification.*

In passing, we note that, unlike the case of Theorem 9, we do know that the converse of Theorem 11 is false: Preference for diversification does not imply pessimism (with respect to c-neutral preferences). This is because the former concept imposes restrictions only on the info-acts that belong to the range of the canonical map φ. The info-acts that are not induced by lotteries lie outside this range, and as a result, a risk preference that exhibits a preference for diversification need not be pessimistic (with respect to c-neutral preferences).

Remark. Dekel (1989) has shown that a continuous (first-order) stochastically increasing preference relation over assets on X is risk averse, provided that it exhibits preference for diversification. Combining this result with Theorem 11, therefore, yields an alternate proof for our Theorem 9.

[24]Here $\simeq$ denotes the symmetric (indifference) part of $\trianglerighteq$.

7 Summary

The ultimate objective of this paper was to introduce a model of evaluating objectively given lotteries in a subjective manner. We have done this by reformulating lotteries as "info-acts," and presuming that a decision maker may view a lottery in her "mind" as such. This model, which we refer to as the *subjective risk theory*, allows the analyst to view one's risk preferences as arising from her preferences over info-acts, thereby relating attitudes toward uncertainty to those toward risk. Our main theoretical finding was that if one is neutral to ambiguity (in her "mind"), then her risk preferences are bound to be of the classical expected utility form. Consequently, this model envisages that any deviation from the expected utility paradigm is to be attributed to one being not neutral toward the evaluation of uncertainty.

The second part of the paper has attempted to investigate the connection between one's attitudes toward uncertainty and the structure of her risk preferences a bit more closely. In particular, we have explored the intuition that if one has a global preference for hedging (in her "mind"), which means that she is rather pessimistic in the evaluation of uncertainty at large, then her (observed) risk preferences must exhibit "pessimism." We have taken this as a definition of pessimism in the risk context (when it is formally possible to do so), and showed that this definition not only matches that used in the literature in the context of RDU theory, but that it yields insights which are not possible to deduce from the latter formulation. In particular, we have shown that the notion of pessimism for risk preferences we introduce here reduces to disappointment aversion in the context of Gul's model, while, in general, it implies probabilistic risk aversion as well as preference for diversification. These properties accord well with what one would intuitively expect from the behavior of a "pessimistic person," and hence provide support for the general formulation we advance here. In general, none of them would be valid with respect to the definition of pessimism utilized in the previous literature.

PROOFS

Notation. For any $\mathbf{p} \in \triangle(X)$ and any map Ψ from $\text{supp}(\mathbf{p})$ into a linear space, we shall adopt the following notation in the remainder of this paper:

$$\textstyle\sum_{\mathbf{p}} \Psi(x) := \displaystyle\sum_{x \in \text{supp}(\mathbf{p})} \Psi(x).$$

PROOF OF PROPOSITION 1

Lemma A.1. Let $\trianglerighteq$ be a preference relation in $\mathfrak{R}(X)$. Then, $\trianglerighteq$ can be represented as in the general weighted EU model if, and only if,

$$\delta_{\max \text{supp}(\mathbf{p})} \trianglerighteq \mathbf{p} \trianglerighteq \delta_{\min \text{supp}(\mathbf{p})} \quad \text{for every } \mathbf{p} \in \triangle(X). \tag{16}$$

Proof. The "only if" part of the assertion is straightforward, so we focus only on its "if" part. Take any $\trianglerighteq \in \mathfrak{R}(X)$ such that (16) holds. For any (fixed) $\mathbf{r}$ in $\triangle(X)$, let $x_*(\mathbf{r})$ and $x^*(\mathbf{r})$ stand for the minimum and maximum elements of the set $\text{supp}(\mathbf{r})$, respectively, and define

$$A := \{\lambda \in [0,1] : \mathbf{r} \trianglerighteq \delta_{(1-\lambda)x_*(\mathbf{r})+\lambda x^*(\mathbf{r})}\} \quad \text{and} \quad B := \{\lambda \in [0,1] : \delta_{(1-\lambda)x_*(\mathbf{r})+\lambda x^*(\mathbf{r})} \trianglerighteq \mathbf{r}\}.$$

These sets are nonempty, for $0 \in A$ and $1 \in B$ by (16). On the other hand, both A and B are closed (because $\trianglerighteq$ is continuous) while $A \cup B = [0,1]$ (because $\trianglerighteq$ is complete). Since $[0,1]$ is connected, therefore, $A \cap B \neq \emptyset$. Conclusion: For every $\mathbf{r} \in \triangle(X)$ there is a $\lambda(\mathbf{r}) \in [0,1]$ such that

$$\mathbf{r} \approx \delta_{(1-\lambda(\mathbf{r}))x_*(\mathbf{r})+\lambda(\mathbf{r})x^*(\mathbf{r})},$$

where $\approx$ is the symmetric part of $\trianglerighteq$. Now define $U : \triangle(X) \to \mathbb{R}$ by

$$U(\mathbf{r}) := (1-\lambda(\mathbf{r}))x_*(\mathbf{r}) + \lambda(\mathbf{r})x^*(\mathbf{r}).$$

Notice that, for any $\mathbf{r} \in \triangle(X)$ we have $U(\mathbf{r}) = \sum_{\mathbf{r}} \pi(x,\mathbf{r})x$, where $\pi(x_*(\mathbf{r}),\mathbf{r}) = 1-\lambda(\mathbf{r})$, $\pi(x^*(\mathbf{r}),\mathbf{r}) = \lambda(\mathbf{r})$, and $\pi(x,\mathbf{r}) = 0$ for any x in $\text{supp}(\mathbf{r})$ distinct from $x_*(\mathbf{r})$ and $x^*(\mathbf{r})$. On the other hand, U represents $\trianglerighteq$. Indeed, for any $\mathbf{p}$ and $\mathbf{q}$ in $\triangle(X)$, we have $\mathbf{p} \trianglerighteq \mathbf{q}$ iff $\delta_{U(\mathbf{p})} \trianglerighteq \delta_{U(\mathbf{q})}$ by definition of U, while the latter statement holds iff $U(\mathbf{p}) \geq U(\mathbf{q})$ because money is a desirable commodity according to $\trianglerighteq$. We conclude that $\trianglerighteq$ admits a general weighted EU representation. ■

To prove Proposition 1, take any preference relation $\succsim$ in $\mathfrak{P}(X)$ and set $\trianglerighteq := \Phi(\succsim)$. As we have verified in Section 2.3, $\trianglerighteq$ belongs to $\mathfrak{R}(X)$. In view of Lemma A.1, therefore, all we need to do is to verify (16) for $\trianglerighteq$. To this end, fix an arbitrary $\mathbf{p}$ in $\triangle(X)$ and let x^* stand for the maximum element of $\text{supp}(\mathbf{p})$. By definition of the maps Φ,

$$\delta_{x^*} \trianglerighteq \mathbf{p} \quad \text{iff} \quad (\langle \delta_{x^*} \rangle, \delta_{x^*}\mathbf{1}_\Omega) \succsim \left(\langle \mathbf{p} \rangle, \textstyle\sum_{\mathbf{p}} \delta_x \mathbf{1}_{\Omega(\mathbf{p},x)}\right) \quad \text{iff} \quad (\langle \mathbf{p} \rangle, \delta_{x^*}\mathbf{1}_\Omega) \succsim \left(\langle \mathbf{p} \rangle, \textstyle\sum_{\mathbf{p}} \delta_x \mathbf{1}_{\Omega(\mathbf{p},x)}\right)$$

where we have invoked the State Invariance Axiom to obtain the second equivalence. Put differently, we have $\delta_{x^*} \trianglerighteq \mathbf{p}$ iff $\delta_{x^*}\mathbf{1}_\Omega \succsim^{\mathbf{p}} f_{\mathbf{p}}$, so, by the monotonicity hypothesis of Proposition 1, we may conclude that $\delta_{x^*} \trianglerighteq \mathbf{p}$. As we can similarly verify that $\mathbf{p} \trianglerighteq \delta_{\min \text{supp}(\mathbf{p})}$ as well, our proof is complete.

SOME PRELIMINARY LEMMATA

To prove the remaining theorems, we will make use of the following preliminary results whose statements make use of the following bit of notation:

Notation. For any lottery $\mathbf{p} \in \triangle(X)$ and any nonempty subset C of $\text{supp}(\mathbf{p})$, we define

$$\Omega(\mathbf{p}, C) := \bigcup_{x \in C} \Omega(\mathbf{p}, x),$$

that is, $\Omega(\mathbf{p}, C)$ is the event in Ω in which $\mathbf{p}$ takes value within C.

Lemma A.2. Let $\succsim$ be a monotonic preference relation in $\mathfrak{P}(X)$ such that

$$(\langle \mathbf{p} \rangle, f) \succsim (\langle \mathbf{p} \rangle, g) \quad \text{implies} \quad (\langle \mathbf{p} \rangle, \alpha f + (1-\alpha) g) \succsim (\langle \mathbf{p} \rangle, g) \tag{17}$$

for every $\mathbf{p} \in \triangle(X)$, $f, g \in \mathcal{F}(\mathbf{p})$ and $0 < \alpha < 1$. Then, for any $(\langle \mathbf{p} \rangle, f) \in \blacktriangle(X)$ and nonempty subset C of $\operatorname{supp}(\mathbf{p})$, we have

$$(\langle \mathbf{p} \rangle, r\mathbf{1}_{\Omega(\mathbf{p},C)} + f\mathbf{1}_{\Omega \backslash \Omega(\mathbf{p},C)}) \succsim (\langle \mathbf{p} \rangle, f), \tag{18}$$

where

$$r := \sum_{x \in C} \frac{\mathbf{p}(x)}{\mathbf{p}(C)} f(x).$$

Proof. In view of the continuity of $\succsim$, it is enough to establish the lemma for $\mathbf{p} \in \triangle(X)$ that takes rational values. Fix then some $\mathbf{p} \in \triangle(X)$ with $\mathbf{p}(x) \in \mathbb{Q}$ for each $x \in \operatorname{supp}(\mathbf{p})$. Let C be a nonempty subset of $\operatorname{supp}(\mathbf{p})$. If C is a singleton, there is nothing to prove, so we assume $|C| \geq 2$. Our proof is by induction on the size of $|C|$.

Let us first consider the case $|C| = 2$, so put $C = \{x', x''\}$ for some distinct $x', x'' \in \operatorname{supp}(\mathbf{p})$. As $\mathbf{p}$ is rational-valued, there are two positive integers (with no common divisors) m and n such that

$$\frac{\mathbf{p}(x')}{\mathbf{p}(x') + \mathbf{p}(x'')} = \frac{m}{n}.$$

(Obviously, $m < n$.) Now pick any real numbers $x_0, ..., x_{n-1}$ in $X \backslash \operatorname{supp}(\mathbf{p})$ and set

$$S := \{x_0, ..., x_{n-1}\} \quad \text{and} \quad S_i := \{x_i, ..., x_{(i+m-1) \bmod n}\}, \; i = 0, ..., n-1.$$

We define $\mathbf{q} \in \triangle(X)$ by

$$\mathbf{q} = \sum_{i=0}^{n-1} \tfrac{1}{n} \mathbf{p}(C) \delta_{x_i} + \sum_{x \in \operatorname{supp}(\mathbf{p}) \backslash C} \mathbf{p}(x) \delta_x.$$

(Note that $\operatorname{supp}(\mathbf{q}) = S \cup (\operatorname{supp}(\mathbf{p}) \backslash C)$ and $\mathbf{q}(S) = \mathbf{p}(C)$.) Next, for each $i = 0, ..., n-1$, put

$$g_i := f(x')\mathbf{1}_{\Omega(\mathbf{q},S_i)} + f(x'')\mathbf{1}_{\Omega(\mathbf{q},S \backslash S_i)} + f\mathbf{1}_{\Omega(\mathbf{q},\operatorname{supp}(\mathbf{p}) \backslash C)}$$

which belongs to $\mathcal{F}(\mathbf{q})$. As $\mathbf{q}(S_i) = \mathbf{p}(x')$ and $\mathbf{q}(S \backslash S_i) = \mathbf{p}(x'')$ for each $i = 0, ..., n-1$, we may invoke the State Invariance Axiom to find

$$(\langle \mathbf{p} \rangle, f) \sim (\langle \mathbf{q} \rangle, g_i) \quad \text{for each } i = 0, ..., n-1. \tag{19}$$

Now set

$$g := \sum_{i=0}^{n-1} \tfrac{1}{n} g_i,$$

and note that $g \in \mathcal{F}(\mathbf{q})$ and $g = \left(\frac{m}{n} f(x') + \frac{n-m}{n} f(x'')\right) \mathbf{1}_{\Omega(\mathbf{q},S)} + f\mathbf{1}_{\Omega \backslash \Omega(\mathbf{q},S)}$. As $\Omega(\mathbf{q}, S) = \Omega(\mathbf{p}, C)$, defining r as in the statement of the lemma allows to write $g = r\mathbf{1}_{\Omega(\mathbf{p},C)} + f\mathbf{1}_{\Omega \backslash \Omega(\mathbf{p},C)}$. Clearly, by the State Invariance Axiom, we have

$$(\langle \mathbf{q} \rangle, g) \sim (\langle \mathbf{p} \rangle, r\mathbf{1}_{\Omega(\mathbf{p},C)} + f\mathbf{1}_{\Omega \backslash \Omega(\mathbf{p},C)}). \tag{20}$$

But as $(\langle \mathbf{q} \rangle, g_0) \sim \cdots \sim (\langle \mathbf{q} \rangle, g_{n-1})$ by (19), applying the hypothesis (17) inductively yields

$$(\langle \mathbf{q} \rangle, g) \succsim (\langle \mathbf{q} \rangle, g_i) \quad \text{for each } i = 0, ..., n-1.$$

Combining this observation with (19) and (20) establishes (18).

As the induction hypothesis, we now assume that our claim is valid for any fixed integer $2 \leq k < |\operatorname{supp}(\mathbf{p})|$ and nonempty subset C of $\operatorname{supp}(\mathbf{p})$ with $|C| \leq k$. To complete the proof, take any $C \subseteq \operatorname{supp}(\mathbf{p})$ with $|C| = k+1$, say, $C = \{x_0, ..., x_k\}$ for some distinct $x_0, ..., x_k \in \operatorname{supp}(\mathbf{p})$. We define $\mathbf{q} \in \triangle(X)$ by

$$\mathbf{q}(x) := \begin{cases} \mathbf{p}(x), & \text{if } x = x_0 \text{ or } x \in \operatorname{supp}(\mathbf{p}) \backslash C \\ \mathbf{p}(C \backslash \{x_0\}), & \text{if } x = x_1. \end{cases}$$

(Note that $\mathbf{p}(C) = \mathbf{q}\{x_0, x_1\}$ and $\mathbf{p}(x_0) = \mathbf{q}(x_0)$.) Next, we put

$$r' := \sum_{i=1}^{k} \frac{\mathbf{p}(x_i)}{\mathbf{p}(C\backslash\{x_0\})} f(x_i),$$

and note that

$$r = \frac{\mathbf{p}(x_0)}{\mathbf{p}(C)} f(x_0) + \frac{\mathbf{p}(C\backslash\{x_0\})}{\mathbf{p}(C)} r',$$

where r is as defined in the statement of the present lemma. Then, first by applying the State Invariance Axiom and then what we have found in the previous paragraph (where $\mathbf{q}$ plays the role of $\mathbf{p}$ and $\{x_0, x_1\}$ that of C), we find

$$\begin{aligned}(\langle\mathbf{p}\rangle, r\mathbf{1}_{\Omega(\mathbf{p},C)} + f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{p},C)}) &\sim (\langle\mathbf{q}\rangle, r\mathbf{1}_{\Omega(\mathbf{q},\{x_0,x_1\})} + f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},\{x_0,x_1\})}) \\ &\succsim (\langle\mathbf{q}\rangle, f(x_0)\mathbf{1}_{\Omega(\mathbf{q},\{x_0\})} + r'\mathbf{1}_{\Omega(\mathbf{q},\{x_1\})} + f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},\{x_0,x_1\})}).\end{aligned}$$

On the other hand, applying the State Invariance Axiom, and then the induction hypothesis (with $C\backslash\{x_0\}$ playing the role of C) yield

$$\begin{aligned}(\langle\mathbf{q}\rangle, f(x_0)\mathbf{1}_{\Omega(\mathbf{q},\{x_0\})} + r'\mathbf{1}_{\Omega(\mathbf{q},\{x_1\})} + f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},\{x_0,x_1\})}) &\sim (\langle\mathbf{p}\rangle, f(x_0)\mathbf{1}_{\Omega(\mathbf{p},\{x_0\})} + r'\mathbf{1}_{\Omega(\mathbf{p},C\backslash\{x_0\})} + f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{p},C)}) \\ &\succsim (\langle\mathbf{p}\rangle, f).\end{aligned}$$

Combining these two findings, we get $(\langle\mathbf{p}\rangle, r\mathbf{1}_{\Omega(\mathbf{p},C)} + f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{p},C)}) \succsim (\langle\mathbf{p}\rangle, f)$, and our proof is complete.

The proof above modifies trivially to show that if (17) is replaced with

$$(\langle\mathbf{p}\rangle, f) \succsim (\langle\mathbf{p}\rangle, g) \quad \text{implies} \quad (\langle\mathbf{p}\rangle, f) \succsim (\langle\mathbf{p}\rangle, \alpha f + (1-\alpha)g)$$

in the statement of Lemma A.2, then (18) would modify to

$$(\langle\mathbf{p}\rangle, f) \succsim (\langle\mathbf{p}\rangle, f\mathbf{1}_{\Omega\backslash\Omega(\mathbf{p},C)} + r\mathbf{1}_{\Omega(\mathbf{p},C)}).$$

Consequently, setting $C = \text{supp}(\mathbf{p})$ in Lemma A.2, adopting the notation

$$\mathbf{p} \odot f := \sum_{x\in\text{supp}(\mathbf{p})} \mathbf{p}(x)f(x) \quad \text{for any } (\langle\mathbf{p}\rangle, f) \in \blacktriangle(X),$$

and using this "dual" observation, we get:

Corollary A.3. Let $\succsim$ be a monotonic preference relation in $\mathfrak{P}(X)$ such that (17) holds for every $\mathbf{p} \in \triangle(X)$, $f, g \in \mathcal{F}(\mathbf{p})$ and $0 < \alpha < 1$. Then,

$$(\langle\mathbf{p}\rangle, (\mathbf{p} \odot f)\,\mathbf{1}_\Omega) \succsim (\langle\mathbf{p}\rangle, f) \tag{21}$$

for each $(\langle\mathbf{p}\rangle, f) \in \blacktriangle(X)$. Moreover, if

$$(\langle\mathbf{p}\rangle, f) \sim (\langle\mathbf{p}\rangle, g) \quad \text{implies} \quad (\langle\mathbf{p}\rangle, \alpha f + (1-\alpha)g) \sim (\langle\mathbf{p}\rangle, f), \tag{22}$$

for every $\mathbf{p} \in \triangle(X)$, $f, g \in \mathcal{F}(\mathbf{p})$ and $0 < \alpha < 1$, then $\sim$ holds in (21).

PROOF OF THEOREM 2

Lemma A.4. Let $\succsim$ be a monotonic and ambiguity neutral preference relation in $\mathfrak{P}(X)$. Then, there is a continuous and strictly increasing function $u : X \to \mathbb{R}$, and for each $\mathbf{p} \in \triangle(X)$, there is a probability measure $\mu_\mathbf{p}$ on the algebra generated by $\Omega(\mathbf{p})$ such that

$$(\langle\mathbf{p}\rangle, f) \succsim (\langle\mathbf{p}\rangle, g) \quad \text{iff} \quad \textstyle\sum_\mathbf{p} \mu_\mathbf{p}(\Omega(\mathbf{p}, x))\mathbb{E}(u, f(x)) \geq \sum_\mathbf{p} \mu_\mathbf{p}(\Omega(\mathbf{p}, x))\mathbb{E}(u, g(x)), \tag{23}$$

for every $f, g \in \mathcal{F}(\mathbf{p})$.

Proof. Ambiguity neutrality of $\succsim$ means that, for every $\mathbf{p} \in \triangle(X)$, there exist two preference relations $\succsim_0^{\mathbf{p}}$ and $\succsim_1^{\mathbf{p}}$ on $\mathcal{F}(\mathbf{p})$ such that

(i) there is a map $u_{\mathbf{p}} : X \to \mathbb{R}$ and a probability measure $\mu_{\mathbf{p}}$ on the algebra generated by $\Omega(\mathbf{p})$ such that the map $V_{\mathbf{p}} : \mathcal{F}(\mathbf{p}) \to \mathbb{R}$, defined by

$$V_{\mathbf{p}}(f) := \textstyle\sum_{\mathbf{p}} \mu_{\mathbf{p}}(\Omega(\mathbf{p}, x))\mathbb{E}(u_{\mathbf{p}}, f(x)),$$

represents $\succsim_0^{\mathbf{p}}$;

(ii) there is a map $u'_{\mathbf{p}} : X \to \mathbb{R}$ and a probability measure $\mu'_{\mathbf{p}}$ on the algebra generated by $\Omega(\mathbf{p})$ such that the map $V_{\mathbf{p}} : \mathcal{F}(\mathbf{p}) \to \mathbb{R}$, defined by

$$V_{\mathbf{p}}(f) := \textstyle\sum_{\mathbf{p}} \mu'_{\mathbf{p}}(\Omega(\mathbf{p}, x))\mathbb{E}(u'_{\mathbf{p}}, f(x)),$$

represents $\succsim_1^{\mathbf{p}}$;

(iii) $r\mathbf{1}_\Omega \succsim_1^{\mathbf{p}} g$ implies $r\mathbf{1}_\Omega \succsim^{\mathbf{p}} g$ and $r\mathbf{1}_\Omega \succsim^{\mathbf{p}} g$ implies $r\mathbf{1}_\Omega \succsim_0^{\mathbf{p}} g$ for each $r \in \triangle(X)$ and $g \in \mathcal{F}(\mathbf{p})$.
As implied by Theorem 17 of Ghirardato and Marinacci (2002), it must be that $u'_{\mathbf{p}}$ is a positive affine transformation of $u_{\mathbf{p}}$, and $\mu'_{\mathbf{p}} = \mu_{\mathbf{p}}$ for each $\mathbf{p} \in \triangle(X)$, meaning that $\succsim_1^{\mathbf{p}}$ coincides with $\succsim_0^{\mathbf{p}}$ for each $\mathbf{p} \in \triangle(X)$.

From (i) we see that $r\mathbf{1}_\Omega \succsim_0^{\mathbf{p}} r'\mathbf{1}_\Omega$ iff $\mathbb{E}(u_{\mathbf{p}}, r) \geq \mathbb{E}(u_{\mathbf{p}}, r')$ for every $r, r' \in \triangle(X)$ and $\mathbf{p} \in \triangle(X)$. On the other hand, by the State Invariance Axiom, $r\mathbf{1}_\Omega \succsim_0^{\mathbf{p}} r'\mathbf{1}_\Omega$ iff $r\mathbf{1}_\Omega \succsim_0^{\mathbf{q}} r'\mathbf{1}_\Omega$, for every $r, r' \in \triangle(X)$ and $\mathbf{p}, \mathbf{q} \in \triangle(X)$. It follows that we may take $u_{\mathbf{p}}$ to be independent of $\mathbf{p}$ in the representation of $\succsim_0^{\mathbf{p}}$. (We thus write u for $u_{\mathbf{p}}$ in what follows.) In turn, (iii) implies that $\delta_x\mathbf{1}_\Omega \succsim^{\mathbf{p}} \delta_y\mathbf{1}_\Omega$ iff $u(x) \geq u(y)$ for every $x, y \in X$, so, as $\succsim$ satisfies the desirability of money, we see that u must be strictly increasing. On the other hand, for any given $\mathbf{p}$, if we define the preference relation $\trianglerighteq$ on $\triangle(X)$ by $r \trianglerighteq r'$ iff $r\mathbf{1}_\Omega \succsim^{\mathbf{p}} r'\mathbf{1}_\Omega$, (i) and (iii) entail that u is a von Neumann-Morgenstern utility function for $\trianglerighteq$. Thus $\trianglerighteq$ satisfies the standard Independence Axiom. Besides, as $\succsim^{\mathbf{p}}$ is continuous, $\trianglerighteq$ is continuous as well. But then, $\trianglerighteq$ admits an expected utility representation with a continuous von Neumann-Morgenstern utility function, and of course, u must then be a positive affine transformation of that function. It follows that u is continuous.

Now fix any $\mathbf{p} \in \triangle(X)$, and take any f and g in $\mathcal{F}(\mathbf{p})$. Given that $\succsim_0^{\mathbf{p}}$ admits an expected utility representation in the standard sense, there exists a lottery r in $\triangle(X)$ such that $f \sim_0^{\mathbf{p}} r\mathbf{1}_\Omega$. By (iii), then, $r\mathbf{1}_\Omega \succsim^{\mathbf{p}} f$. To derive a contradiction, suppose $r\mathbf{1}_\Omega \succ^{\mathbf{p}} f$. Then, by monotonicity of $\succsim^{\mathbf{p}}$, there must be an $x \in \text{supp}(\mathbf{p})$ such that $r\mathbf{1}_\Omega \succ^{\mathbf{p}} f(x)\mathbf{1}_\Omega$. In turn, by continuity of $\succsim^{\mathbf{p}}$, we can find a (large enough) λ in $(0,1)$ with $r\mathbf{1}_\Omega \succ^{\mathbf{p}} (\lambda r + (1-\lambda)f(x))\mathbf{1}_\Omega \succ^{\mathbf{p}} f$. But then, by (iii), we find $r\mathbf{1}_\Omega \succ_0^{\mathbf{p}} (\lambda r + (1-\lambda)f(x))\mathbf{1}_\Omega \succsim_0^{\mathbf{p}} f$, that is, $r\mathbf{1}_\Omega \succ_0^{\mathbf{p}} f$, a contradiction. Conclusion: $f \sim^{\mathbf{p}} r\mathbf{1}_\Omega$. Consequently, using (iii) again,

$$f \succsim_0^{\mathbf{p}} g \quad \text{iff} \quad r\mathbf{1}_\Omega \succsim_0^{\mathbf{p}} g \quad \text{iff} \quad r\mathbf{1}_\Omega \succsim^{\mathbf{p}} g \quad \text{iff} \quad f \succsim^{\mathbf{p}} g.$$

Combining this with (i), and recalling that $u_{\mathbf{p}}$ does not depend on $\mathbf{p}$, completes the proof.

To prove Theorem 2, let $\succsim$ be a monotonic and ambiguity neutral element of $\mathfrak{P}(X)$, and find $u : X \to \mathbb{R}$ and $\mu_{\mathbf{p}}$ (for each $\mathbf{p}$ in $\triangle(X)$) as in Lemma A.4. Note in particular that, for any $\mathbf{p} \in \triangle(X)$,

$$r\mathbf{1}_\Omega \succsim^{\mathbf{p}} r'\mathbf{1}_\Omega \quad \text{iff} \quad \mathbb{E}(u, r) \geq \mathbb{E}(u, r') \tag{24}$$

for every $r, r' \in \triangle(X)$. Another consequence of the representation (23) is that, for any $\mathbf{p} \in \triangle(X)$, the preference relation $\succsim^{\mathbf{p}}$ on $\mathcal{F}(\mathbf{p})$ satisfies (22), and hence, by Corollary A.3,

$$(\langle \mathbf{p} \rangle, f) \sim (\langle \mathbf{p} \rangle, (\mathbf{p} \odot f)\, \mathbf{1}_\Omega) \qquad \text{for every } f \in \mathcal{F}(\mathbf{p}).$$

But then, for any $(\langle \mathbf{p} \rangle, f)$ and $(\langle \mathbf{q} \rangle, g)$ in $\blacktriangle(X)$, we have $(\langle \mathbf{p} \rangle, f) \succsim (\langle \mathbf{q} \rangle, g)$ iff

$$(\langle \mathbf{p} \rangle, (\mathbf{p} \odot f)\, \mathbf{1}_\Omega) \succsim (\langle \mathbf{q} \rangle, (\mathbf{q} \odot g)\, \mathbf{1}_\Omega),$$

which, by the State Invariance Axiom, holds iff

$$(\langle \mathbf{p} \rangle, (\mathbf{p} \odot f)\, \mathbf{1}_\Omega) \succsim (\langle \mathbf{p} \rangle, (\mathbf{q} \odot g)\, \mathbf{1}_\Omega),$$

which, in view of (24), holds iff $\mathbb{E}(u, \mathbf{p} \odot f) \geq \mathbb{E}(u, \mathbf{q} \odot g)$, that is,

$$\textstyle\sum_{\mathbf{p}} \mathbf{p}(x)\mathbb{E}(u, f(x)) \geq \sum_{\mathbf{p}} \mathbf{q}(x)\mathbb{E}(u, g(x)).$$

It follows that, for any $\mathbf{p}, \mathbf{q} \in \triangle(X)$, we have $\varphi(\mathbf{p}) \succsim \varphi(\mathbf{q})$ iff $\mathbb{E}(u, \mathbf{p}) \geq \mathbb{E}(u, \mathbf{q})$, and Theorem 2 is proved.

PROOF OF PROPOSITION 3

Nomenclature. For any lottery $r \in \triangle(X)$, a *certainty equivalent* of r relative to a map $u : X \to \mathbb{R}$ is defined as a number x in X with $\mathbb{E}(u, r) = u(x)$. If u is continuous and strictly increasing, then such a number exists and it is unique; in this case we denote it by $\mathrm{ce}(u, r)$.

Lemma A.5. Let $\succsim$ be a monotonic preference relation in $\mathfrak{P}(X)$ such that there is biseparable EU function V that represents $\succsim^{\mathbf{q}}$ for some $\mathbf{q}$ in $\triangle(X)$. Then,

$$(\langle \mathbf{p} \rangle, f) \sim \left(\langle \mathbf{p} \rangle, \textstyle\sum_{\mathbf{p}} \delta_{\mathrm{ce}(u, f(x))} \mathbf{1}_{\Omega(\mathbf{p}, x)} \right) \qquad \text{for every } (\langle \mathbf{p} \rangle, f) \in \blacktriangle(X), \tag{25}$$

where $u : X \to \mathbb{R}$ is defined by $u(x) := V(\delta_x \mathbf{1}_\Omega)$.

Proof. Take any $(\langle \mathbf{p} \rangle, f)$ in $\blacktriangle(X)$, and fix any $\omega \in \Omega$. The State Invariance Axiom guarantees that any two info-acts whose acts are constant and the same are indifferent relative to $\succsim$. Furthermore, as V is a biseparable EU function, we have $V(f(x)\mathbf{1}_\Omega) = \mathbb{E}(u, f(x)) = u(\mathrm{ce}(u, f(x))) = V(\delta_{\mathrm{ce}(u,f(x))} \mathbf{1}_\Omega)$ for every x in $\mathrm{supp}(\mathbf{p})$. Hence, as V represents $\succsim^{\mathbf{q}}$, we have $(\langle \mathbf{q} \rangle, f(x)\mathbf{1}_\Omega) \sim (\langle \mathbf{q} \rangle, \delta_{\mathrm{ce}(u,f(x))} \mathbf{1}_\Omega)$. Combining these observations, we find

$$(\langle \mathbf{p} \rangle, f(x)\mathbf{1}_\Omega) \sim (\langle \mathbf{q} \rangle, f(x)\mathbf{1}_\Omega) \sim (\langle \mathbf{q} \rangle, \delta_{\mathrm{ce}(u,f(x))} \mathbf{1}_\Omega) \sim (\langle \mathbf{p} \rangle, \delta_{\mathrm{ce}(u,f(x))} \mathbf{1}_\Omega)$$

for each $x \in \mathrm{supp}(\mathbf{p})$, and (25) follows from the monotonicity of $\succsim^{\mathbf{p}}$.

To prove Proposition 3, let $\succsim_1$ and $\succsim_2$ be two preference relations in $\mathfrak{P}(X)$ that have biseparable EU representations such that $\succsim_1 |\blacktriangle_{\mathrm{d}}(X) = \succsim_2 |\blacktriangle_{\mathrm{d}}(X)$. Fix any real number α with $0 < \alpha < 1$, and define

$$A := \{(x, y) \in \mathrm{int}(X) \times \mathrm{int}(X) : x \geq y\}.$$

Pick any $(a, b) \in A$, consider the lottery $\mathbf{q} \in \triangle(X)$ defined as $\mathbf{q} := \alpha\delta_a + (1 - \alpha)\delta_b$, and set $S := \Omega(\mathbf{q}, a)$. For any $(x, y) \in A$, we define the map $f_{x,y} \in \mathcal{F}(\mathbf{q})$ as

$$f_{x,y} := \delta_x \mathbf{1}_S + \delta_y \mathbf{1}_{\Omega \setminus S}.$$

Next, for each $i \in \{1, 2\}$, we define the preference relation $\trianglerighteq_i$ on A as

$$(x, y) \trianglerighteq_i (z, w) \quad \text{ iff } \quad (\langle \mathbf{q} \rangle, f_{x,y}) \succsim_i (\langle \mathbf{q} \rangle, f_{z,w}).$$

For each i, the biseparable EU representability of $\succsim_i$ ensures that there is a real number $\theta_i \in [0, 1]$ and a continuous map $V_i : \mathcal{F}(\mathbf{q}) \to \mathbb{R}$ such that, where $u_i : X \to \mathbb{R}$ is defined by $u_i(t) := V_i(\delta_t \mathbf{1}_\Omega)$, the map $(x, y) \mapsto \theta_i u_i(x) + (1 - \theta_i)u_i(y)$ represents $\trianglerighteq_i$ and we have $V_i(\mathbf{p}\mathbf{1}_\Omega) = \mathbb{E}(u_i, \mathbf{p})$ for every $\mathbf{p} \in \triangle(X)$. But, as $\succsim_1$ and $\succsim_2$ agree on degenerate info-acts, we have $\trianglerighteq_1 = \trianglerighteq_2$, and it follows that the maps $(x, y) \mapsto \theta_1 u_1(x) + (1 - \theta_1)u_1(y)$ and $(x, y) \mapsto \theta_2 u_2(x) + (1 - \theta_2)u_2(y)$ represent the same preference relation on A. (Furthermore, as $(\langle \mathbf{q} \rangle, \delta_x \mathbf{1}_\Omega) \succsim_i (\langle \mathbf{q} \rangle, \delta_y \mathbf{1}_\Omega)$ iff $x \geq y$, for every $x, y \in X$, the map u_i must be strictly increasing on X.) By Theorem 3.2 of Wakker (1993), then, there exist real numbers τ_1, τ_2 and β such that $\beta > 0$,

$$\theta_1 u_1(x) = \beta\theta_2 u_2(x) + \tau_1 \quad \text{ and } \quad (1 - \theta_1)u_1(x) = \beta(1 - \theta_2)u_2(x) + \tau_2$$

for every x in $\mathrm{int}(X)$. It follows that u_1 is a positive affine transformation of u_2 on $\mathrm{int}(X)$, and by continuity, on the entire X, namely, $u_1 = \beta u_2 + (\tau_1 + \tau_2)$. Consequently, $\mathrm{ce}(u_1, r) = \mathrm{ce}(u_2, r)$ for every $r \in \triangle(X)$. Combining this fact with Lemma A.5, we find that

$$(\langle \mathbf{p} \rangle, f) \sim_1 \left(\langle \mathbf{p} \rangle, \textstyle\sum_{\mathbf{p}} \delta_{\mathrm{ce}(u_1, f(x))} \mathbf{1}_{\Omega(\mathbf{p}, x)} \right) \sim_2 (\langle \mathbf{p} \rangle, f)$$

for every $(\langle \mathbf{p}\rangle, f) \in \blacktriangle(X)$. As $\succsim_1$ and $\succsim_2$ agree on degenerate info-acts, therefore, it follows from this observation that they must agree on the entire $\blacktriangle(X)$. Proposition 3 is now proved.

PROOF OF PROPOSITION 5

For any $\mathbf{p} \in \triangle(X)$, let $\mathcal{A}(\mathbf{p})$ stand for the algebra generated by $\Omega(\mathbf{p})$, and define $\nu_{\mathbf{p}} : \mathcal{A}(\mathbf{p}) \to [0,1]$ by

$$\nu_{\mathbf{p}}(A) := w(\langle \mathbf{p}\rangle(A)).$$

It is plain that $\nu_{\mathbf{p}}$ is a normalized capacity (that is, a monotonic set function that assigns 0 to $\emptyset$ and 1 to Ω) on $\mathcal{A}(\mathbf{p})$. We define $V : \blacktriangle(X) \to \mathbb{R}$ by

$$V(\langle \mathbf{p}\rangle, f) := \int_X \mathbb{E}(u, f(x))\, \nu_{\mathbf{p}}(dx)$$

where the integral in the expression is understood as the Choquet integral with respect to $\nu_{\mathbf{p}}$. Let $\succsim$ be the preference relation on $\blacktriangle(X)$ which is represented by V. It is readily checked that $\succsim$ is a biseparable EU preference on $\blacktriangle(X)$ such that $\Phi(\succsim) = \trianglerighteq$.

Suppose that w is a convex function, and fix an arbitrary $\mathbf{p} \in \triangle(X)$. Then, for any A and B in $\mathcal{A}(\mathbf{p})$, the increasing difference property of convex functions entails that

$$\begin{aligned}
\nu_{\mathbf{p}}(B) - \nu_{\mathbf{p}}(A \cap B) &= w(\langle \mathbf{p}\rangle(A)) - w(\langle \mathbf{p}\rangle(A \cap B)) \\
&= w\left(\langle \mathbf{p}\rangle(A \cap B) + \langle \mathbf{p}\rangle(B \backslash A)\right) - w(\langle \mathbf{p}\rangle(A \cap B)) \\
&\leq w(\langle \mathbf{p}\rangle(A) + \langle \mathbf{p}\rangle(B \backslash A)) - w(\langle \mathbf{p}\rangle(A)) \\
&= \nu_{\mathbf{p}}(A \cup B) - \nu_{\mathbf{p}}(A).
\end{aligned}$$

Conclusion: $\nu_{\mathbf{p}}$ is a convex capacity. By the Proposition in Section 3 of Schmeidler (1989), we may thus conclude that $\succsim^{\mathbf{p}}$ is uncertainty averse.

Conversely, suppose $\succsim$ is uncertainty averse. Then by the Proposition of Schmeidler (1989) we have just mentioned, $\nu_{\mathbf{p}}$ is a convex capacity for each $\mathbf{p} \in \triangle(X)$. Take any s and t in $(0,1)$, and say, $s < t$. Then, consider the lottery $\mathbf{p} \in \triangle(X)$ defined as

$$\mathbf{p} = s\delta_{x_1} + \frac{t-s}{2}\delta_{x_2} + \frac{t-s}{2}\delta_{x_3} + (1-t)\delta_{x_4},$$

where $x_1, ..., x_4$ are arbitrarily chosen numbers in X with $x_1 < x_2 < x_2 < x_4$. Then, where $A = \Omega(\mathbf{p}, \{x_1, x_2\})$ and $B = \Omega(\mathbf{p}, \{x_1, x_3\})$, we have

$$2w\left(\tfrac{1}{2}s + \tfrac{1}{2}t\right) = \nu_{\mathbf{p}}(A) + \nu_{\mathbf{p}}(B) \leq \nu_{\mathbf{p}}(A \cap B) + \nu_{\mathbf{p}}(A \cup B) = w(s) + w(t).$$

In view of the arbitrariness of s and t, and the continuity of w, we may thus conclude that w is a convex function. Proposition 5 is now proved.

PROOF OF PROPOSITION 6

Let $\trianglerighteq$ be represented by (β, u) as in the Gul model. Let $\succsim$ be a preference relation over $\blacktriangle(X)$ that is represented by a map $V : \blacktriangle(X) \to \mathbb{R}$ with

$$V(\langle \mathbf{p}\rangle, f) := \textstyle\sum_{\mathbf{p}} \lambda_{\mathbf{p},f}(x)\mathbf{p}(x)\mathbb{E}(u, f(x)), \tag{26}$$

where

$$\lambda_{\mathbf{p},f}(x) := \begin{cases} \frac{1+\beta}{1+\beta-\alpha_{\mathbf{p},f}\beta} & \text{if } \mathbb{E}(u, f(x)) < V(\langle \mathbf{p}\rangle, f), \\ \frac{1}{1+\beta-\alpha_{p,f}\beta} & \text{if } \mathbb{E}(u, f(x)) \geq V(\langle \mathbf{p}\rangle, f), \end{cases}$$

and

$$\alpha_{\mathbf{p},f} = \mathbf{p}(\{x \in \operatorname{supp}(\mathbf{p}) : \mathbb{E}(u, f(x)) \geq V(\langle \mathbf{p}\rangle, f)\}).$$

In comparison with (13), $\alpha_{\mathbf{p},f}$ is not present in the numerators of $\lambda_{\mathbf{p},f}(x)$ because they are multiplied here by the unconditional probability $\mathbf{p}(x)$ rather than the probability conditional on x belonging to either disappointment or elation part of $\mathbf{p}$. It is routine to verify that $\succsim \in \mathfrak{P}(X)$ and that $\succsim$ is a biseparable EU preference on $\blacktriangle(X)$. Clearly, $\Phi(\succsim) = \trianglerighteq$.

Suppose $\beta \geq 0$. Fix an arbitrary $\mathbf{p} \in \triangle(X)$, $f, g \in \mathcal{F}(\mathbf{p})$, $\alpha \in (0,1)$, and let $g \succsim^{\mathbf{p}} f$. As suggested by Gul (1991, p. 674), the upper contour set of the preference relation $\succsim^{\mathbf{p}}$ at $f \in \mathcal{F}(\mathbf{p})$, defined as $U_{\mathbf{p},f} := \{h \in \mathcal{F}(\mathbf{p}) : h \succsim^{\mathbf{p}} f\}$, can be computed as the set of all $h \in \mathcal{F}(\mathbf{p})$ that satisfy

$$V^{\mathrm{loc}}(\langle \mathbf{p} \rangle, h; v) := \textstyle\sum_{\mathbf{p}} \left(\dfrac{1}{1+\beta} \mathbb{E}(u, h(x)) + \dfrac{\beta}{1+\beta} \min \{\mathbb{E}(u, h(x)), v\} \right) \mathbf{p}(x) \geq v,$$

where $v = V(\langle \mathbf{p} \rangle, f)$. Moreover, this inequality holds as equality if $h \sim^{\mathbf{p}} f$. Thus, we have

$$\textstyle\sum_{\mathbf{p}} \left(\dfrac{1}{1+\beta} \mathbb{E}(u, g(x)) + \dfrac{\beta}{1+\beta} \min \{\mathbb{E}(u, g(x)), v\} \right) \mathbf{p}(x) \geq v,$$

and

$$\textstyle\sum_{\mathbf{p}} \left(\dfrac{1}{1+\beta} \mathbb{E}(u, f(x)) + \dfrac{\beta}{1+\beta} \min \{\mathbb{E}(u, f(x)), v\} \right) \mathbf{p}(x) = v.$$

Consequently,

$$\textstyle\sum_{\mathbf{p}} \left(\dfrac{1}{1+\beta} \mathbb{E}(u, \alpha f(x) + (1-\alpha) g(x)) + \dfrac{\beta}{1+\beta} \min \{\mathbb{E}(u, \alpha f(x) + (1-\alpha) g(x)), v\} \right) \mathbf{p}(x) \geq v$$

because $\mathbb{E}(u, \cdot)$ is a linear function and $\min(\cdot, v)$ is concave. Therefore, $\alpha f + (1-\alpha) g \succsim^{\mathbf{p}} f$, that is, $\succsim^{\mathbf{p}}$ is uncertainty averse.

Conversely, suppose $\succsim^{\mathbf{p}}$ is uncertainty averse for every $\mathbf{p} \in \triangle(X)$. We are to show that $\beta \geq 0$. To derive a contradiction, suppose $\beta < 0$. Let $\mathbf{p} := (-\beta)\mathbf{1}_a + (1+\beta)\mathbf{1}_b$ for some distinct a and b in X, and fix some $v \in \mathrm{int}(u(X))$. Consider two acts $h_1, h_2 \in \mathcal{F}(\mathbf{p})$ such that

$$\begin{aligned} \mathbb{E}(u, h_1(a)) &= v - \varepsilon, & \mathbb{E}(u, h_2(a)) &= v + \frac{2(1+\beta)^2}{(-\beta)} \varepsilon, \\ \mathbb{E}(u, h_1(b)) &= v - \beta\varepsilon, & \mathbb{E}(u, h_2(b)) &= v - 2\varepsilon, \end{aligned}$$

where $\varepsilon > 0$ is sufficiently small so that the values in the right-hand side of these equations are in $u(X)$, and, therefore, acts h_1 and h_2 exist. Evaluating $V^{\mathrm{loc}}(\langle \mathbf{p} \rangle, h_1; v)$ and $V^{\mathrm{loc}}(\langle \mathbf{p} \rangle, h_2; v)$, we observe that

$$\begin{aligned} V^{\mathrm{loc}}(\langle \mathbf{p} \rangle, h_1; v) &= (v - \varepsilon)(-\beta) + \left(\frac{1}{1+\beta}(v - \beta\varepsilon) + \frac{\beta}{1+\beta} v \right)(1+\beta) \\ &= (v - \varepsilon)(-\beta) + v - \beta\varepsilon + \beta v = v, \end{aligned}$$

and

$$\begin{aligned} V^{\mathrm{loc}}(\langle \mathbf{p} \rangle, h_2; v) &= \left(\frac{1}{1+\beta} \left(v + \frac{2(1+\beta)^2}{(-\beta)} \varepsilon \right) + \frac{\beta}{1+\beta} v \right)(-\beta) + (v - 2\varepsilon)(1+\beta) \\ &= \left(v + \frac{2(1+\beta)}{(-\beta)} \varepsilon \right)(-\beta) + (v - 2\varepsilon)(1+\beta) \\ &= v(-\beta) + v(1+\beta) \\ &= v. \end{aligned}$$

Therefore, $h_1 \sim^{\mathbf{p}} f \sim^{\mathbf{p}} h_2$, where $f \in \mathcal{F}(\mathbf{p})$ is such that $V(\langle \mathbf{p} \rangle, f) = v$. (For instance, $f = \delta_{u^{-1}(v)} \mathbf{1}_{\Omega}$.) Now, we choose a number $0 < \alpha < 1$ such that

$$\frac{2(1+\beta)^2}{(-\beta)} < \frac{\alpha}{1-\alpha} < \frac{2}{(-\beta)},$$

which is possible because $\beta < 0$. This choice of α entails that

$$\begin{aligned} \mathbb{E}(u, \alpha h_1(a) + (1-\alpha) h_2(a)) &= \alpha(v - \varepsilon) + (1-\alpha)\left(v + \frac{2(1+\beta)^2}{(-\beta)} \varepsilon \right) \\ &= v + \left(-\alpha + (1-\alpha) \frac{2(1+\beta)^2}{(-\beta)} \right) \varepsilon \end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(u, \alpha h_1(b) + (1-\alpha)h_2(b)) &= \alpha(v - \beta\varepsilon) + (1-\alpha)(v - 2\varepsilon) \\ &= v + (\alpha(-\beta) - 2(1-\alpha))\varepsilon\end{aligned}$$

so that $\mathbb{E}(u, \alpha h_1(a) + (1-\alpha)h_2(a)) < v$ and $\mathbb{E}(u, \alpha h_1(b) + (1-\alpha)h_2(b)) < v$. It follows that

$$V^{\text{loc}}(\langle \mathbf{p} \rangle, \alpha h_1 + (1-\alpha)h_2; v) < v\mathbf{p}(a) + v\mathbf{p}(b) = v,$$

which implies $h_1 \sim^{\mathbf{p}} h_2 \succ^{\mathbf{p}} \alpha h_1 + (1-\alpha)h_2$, contradicting that $\succsim^{\mathbf{p}}$ is uncertainty averse. Proposition 6 is now proved.

PROOF OF PROPOSITION 7

Consider the map $U^* : \blacktriangle(X) \to \mathbb{R}$ defined by (15). It is readily checked that $\trianglerighteq = \Phi(\succsim)$ where $\succsim$ is the preference relation on $\blacktriangle(X)$ which is represented by U^*. Suppose that $\trianglerighteq$ is risk averse. Then, by Theorem 3 of Cerreia-Vioglio, Dillenberger and Ortoleva (2013), there is a subset $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{W}$ is a cautious EU representation for $\trianglerighteq$ and every element of $\mathcal{W}$ is concave. We wish to prove that $\succsim$ is uncertainty averse. To this end, fix an arbitrary $\mathbf{p} \in \triangle(X)$, and take any $f, g \in \mathcal{F}(\mathbf{p})$ with $f \sim^{\mathbf{p}} g$, and any λ in the interval $[0,1]$. Then,

$$\begin{aligned}U^*(\langle \mathbf{p} \rangle, \lambda f + (1-\lambda)g) &= \inf_{v \in \mathcal{W}} v^{-1}\left(\textstyle\sum_{\mathbf{p}} \mathbf{p}(x)v(\lambda \mathbb{E}(f(x)) + (1-\lambda)\mathbb{E}(g(x)))\right) \\ &\geq \inf_{v \in \mathcal{W}} v^{-1}\left(\lambda \textstyle\sum_{\mathbf{p}} \mathbf{p}(x)v(\mathbb{E}(f(x))) + (1-\lambda)\sum_{\mathbf{p}} \mathbf{p}(x)v(\mathbb{E}(g(x)))\right)\end{aligned}$$

because each v in $\mathcal{W}$ is a concave function with a strictly increasing inverse. On the other hand, by definition of U^*, we have $v^{-1}\left(\sum_{\mathbf{p}} \mathbf{p}(x)v(\mathbb{E}(h(x)))\right) \geq U^*(\langle \mathbf{p} \rangle, h)$, and hence,

$$\textstyle\sum_{\mathbf{p}} \mathbf{p}(x)v(\mathbb{E}(h(x))) \geq v(U^*(\langle \mathbf{p} \rangle, h)),$$

for every $v \in \mathcal{W}$ and $h \in \{f, g\}$. Combining this fact with the previous inequality, and recalling that each v in $\mathcal{W}$ has a strictly increasing inverse, then,

$$\begin{aligned}U^*(\langle \mathbf{p} \rangle, \lambda f + (1-\lambda)g) &\geq \inf_{v \in \mathcal{W}} v^{-1}\left(\lambda v(U^*(\langle \mathbf{p} \rangle, f)) + (1-\lambda)v(U^*(\langle \mathbf{p} \rangle, g))\right) \\ &= \inf_{v \in \mathcal{W}} v^{-1}\left(v(U^*(\langle \mathbf{p} \rangle, f))\right) \\ &= U^*(\langle \mathbf{p} \rangle, f),\end{aligned}$$

where the first equality follows from the hypothesis that $f \sim^{\mathbf{p}} g$ (which means $U^*(\langle \mathbf{p} \rangle, f) = U^*(\langle \mathbf{p} \rangle, g)$). In view of the arbitrariness of $\mathbf{p}$, f, g and λ, we may thus conclude that $\succsim$ is uncertainty averse, which proves the "if" part of Proposition 7. We will turn to the proof of the "only if" part of this proposition after Theorem 9 is proved.

PROOF OF PROPOSITION 8

Let $\trianglerighteq$ be represented by (β, u) as in the Gul model. (It is without loss of generality to take u as the utility function used in the representation of $\trianglerighteq$ as in the general weighted EU model.)

[(ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii)] If $\beta = 0$, then $\trianglerighteq$ is an expected utility preference relation, and $\pi(x, \mathbf{p}) = \mathbf{p}(x) = \mathbf{q}(x) = \pi(x, \mathbf{q})$ for any $\mathbf{p}, \mathbf{q} \in \triangle(X)$, and $x \in \text{supp}(\mathbf{p}) \cap \text{supp}(\mathbf{q})$, such that $\mathbf{p}(x) = \mathbf{q}(x)$.

[(i) $\Rightarrow$ (ii)] Assume that $\trianglerighteq$ is π-pessimistic, but to derive a contradiction, suppose $\beta \neq 0$. Let us first consider the case $\beta > 0$. Since u is continuous, $u(X)$ is connected. Given that u is unique up to affine transformations, we may thus assume without loss of generality that $[0,2] \subseteq u(X)$. Pick any $x_0, x_1 \in X$ with $u(x_0) = 0$ and $u(x_1) = 1$, let $y \in X$ be such that $u(y) = \frac{2}{3}$ and let $z \in X$ be such that $u(z) = \frac{1}{3+3\beta}$. Consider the lotteries

$\mathbf{p} = \frac{1}{2}\delta_{x_1} + \frac{1}{4}\delta_y + \frac{1}{4}\delta_{x_0}$ and $\mathbf{q} = \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_z + \frac{1}{2}\delta_{x_0}$. It can be shown that the evaluations of these two lotteries according to (13) are:

$$U(\mathbf{p}) = \left(\tfrac{1}{1+\beta-\alpha_{\mathbf{p}}\beta}\right)\mathbf{p}(x_1) + \left(\tfrac{1}{1+\beta-\alpha_{\mathbf{p}}\beta}\right)\mathbf{p}(y)\tfrac{2}{3} \quad \text{and} \quad U(\mathbf{q}) = \left(\tfrac{1}{1+\beta-\alpha_{\mathbf{q}}\beta}\right)\mathbf{p}(x_1) + \left(\tfrac{1+\beta}{1+\beta-\alpha_{\mathbf{q}}\beta}\right)\mathbf{p}(z)\tfrac{1}{3+3\beta}$$

where $\alpha_{\mathbf{p}} = \frac{3}{4}$ and $\alpha_{\mathbf{q}} = \frac{1}{4}$. To verify that these expressions indeed solve the circular relationships between valuations U and the coefficients, it is sufficient to observe that

$$U(\mathbf{p}) = \frac{2}{3\left(1+\frac{1}{4}\beta\right)} < \tfrac{2}{3} = u(y) \quad \text{and} \quad U(\mathbf{q}) = \frac{1}{3\left(1+\frac{3}{4}\beta\right)} > \frac{1}{3+3\beta} = u(z)$$

for all $\beta > 0$. Therefore, the outcomes that bring elation in $\mathbf{p}$ are x_1 and y, whereas the only outcome that brings elation in $\mathbf{q}$ is x_1, and hence the values of $\alpha_{\mathbf{p}}$ and $\alpha_{\mathbf{q}}$ that we assumed are correct.

Now we observe that the $\mathbf{p}$-rank of y is $\frac{1}{2}$, $\mathbf{q}$-rank of z is $\frac{1}{4}$, and $\mathbf{p}(y) = \mathbf{q}(z)$, but the weights in the general weighted EU form are ranked as

$$\pi(y,\mathbf{p}) = \frac{1}{4+\beta} < \frac{1+\beta}{4+3\beta} = \pi(z,\mathbf{q})$$

for all $\beta > 0$. Finally, consider the lottery $\mathbf{q}' = \frac{1}{4}\delta_{x_1'} + \frac{1}{4}\delta_y + \frac{1}{2}\delta_{x_0'}$, where x_0' and x_1' are such that $u(x_0') = u(y) - u(z)$ and $u(x_1') = 1 + u(y) - u(z)$. Observe that the $\mathbf{q}'$-rank of y is equal to the $\mathbf{q}$-rank of z and yet $\pi(y,\mathbf{q}') = \pi(z,\mathbf{q})$, a contradiction.

Let us now consider the case $-1 < \beta < 0$, and put $\mathbf{p} := \frac{1}{2}\delta_{x_1} + \frac{1}{4}\delta_y + \frac{1}{4}\delta_{x_0}$ and $\mathbf{q} := \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_y + \frac{1}{2}\delta_{x_0}$, where $x_0, x_1, y \in X$ are such that $u(x_0) = 0$, $u(x_1) = 1$ and $u(y) = \frac{1}{3}$. It can then be shown that the evaluations of these two lotteries according to (13) are:

$$U(\mathbf{p}) = \left(\tfrac{1}{1+\beta-\alpha_{\mathbf{p}}\beta}\right)\mathbf{p}(x_1) + \left(\tfrac{1+\beta}{1+\beta-\alpha_{\mathbf{p}}\beta}\right)\mathbf{p}(y)\tfrac{1}{3} \quad \text{and} \quad U(\mathbf{q}) = \left(\tfrac{1}{1+\beta-\alpha_{\mathbf{q}}\beta}\right)\mathbf{p}(x_1) + \left(\tfrac{1+\beta}{1+\beta-\alpha_{\mathbf{q}}\beta}\right)\mathbf{p}(z)\tfrac{1}{3}$$

where $\alpha_{\mathbf{p}} = \frac{1}{2}$ and $\alpha_{\mathbf{q}} = \frac{1}{4}$. To verify that these expressions indeed solve the circular relationships between valuations U and the coefficients, it sufficient to observe that

$$U(\mathbf{p}) = \frac{7+\beta}{12+6\beta} > \tfrac{1}{3} = u(y) \quad \text{and} \quad U(\mathbf{q}) = \frac{4+\beta}{12+9\beta} > \frac{1}{3} = u(y)$$

for all $\beta \in (-1, 0)$. Therefore, the only outcome that brings elation in $\mathbf{p}$ and $\mathbf{q}$ is x_1, and the values of $\alpha_{\mathbf{p}}$ and $\alpha_{\mathbf{q}}$ that we assumed are correct. Observe next that the $\mathbf{p}$-rank order of y is $\frac{1}{2}$, the $\mathbf{q}$-rank of y is $\frac{1}{4}$, and $\mathbf{p}(y) = \mathbf{q}(y)$, but the weights in the general weighted EU form are ranked as

$$\pi(y,\mathbf{p}) = \frac{1+\beta}{4+2\beta} < \frac{1+\beta}{4+\beta} = \pi(y,\mathbf{q})$$

for all $\beta \in (-1, 0)$, a contradiction.

[(iii) $\Rightarrow$ (ii)]. This part can be proved analogously to the previous implication by choosing lotteries $\mathbf{p} = \frac{1}{2}\delta_{x_1} + \frac{1}{4}\delta_y + \frac{1}{4}\delta_{x_0}$ and $\mathbf{q} = \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_y + \frac{1}{2}\delta_{x_0}$, where $x_0, x_1, y \in X$ are such that $u(x_0) = 0$, $u(x_1) = 1$, $u(y) = \frac{2}{3}$ in the case $\beta > 0$, and $\mathbf{p} = \frac{1}{2}\delta_{x_1} + \frac{1}{4}\delta_y + \frac{1}{4}\delta_{x_0}$, $\mathbf{q} = \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_z + \frac{1}{2}\delta_{x_0}$, where $x_0, x_1, y, z \in X$ are such that $u(x_0) = 0$, $u(x_1) = 1$, $u(y) = \frac{3+\beta}{4+2\beta}$, $u(z) = \frac{1}{3}$ in the case $\beta \in (-1, 0)$. Proposition 8 is now proved.

PROOF OF THEOREM 9

Assume that $\succsim$ is uncertainty averse. Then, $\succsim$ satisfies (17) for every $\mathbf{q} \in \Delta(X)$, $f, g \in \mathcal{F}(\mathbf{q})$ and $0 < \alpha < 1$. By Lemma A.2, therefore,

$$\left(\langle\mathbf{q}\rangle, \left(\sum_{x\in C}\frac{\mathbf{q}(x)}{\mathbf{q}(C)}f(x)\right)\mathbf{1}_{\Omega(\mathbf{q},C)} + f\mathbf{1}_{\Omega\setminus\Omega(\mathbf{q},C)}\right) \succsim (\langle\mathbf{q}\rangle, f), \tag{27}$$

for every $(\langle\mathbf{q}\rangle, f) \in \blacktriangle(X)$ and nonempty subset C of $\mathrm{supp}(\mathbf{q})$.

Now take any $\mathbf{p}$ and $\mathbf{q}$ in $\Delta(X)$ such that $\mathbf{q}$ is a mean-preserving spread of $\mathbf{p}$, and put

$$f_{\mathbf{q}} := \sum_{\mathbf{q}} \delta_x \mathbf{1}_{\Omega(\mathbf{q},x)} \quad \text{and} \quad f_{\mathbf{p}} := \sum_{\mathbf{p}} \delta_x \mathbf{1}_{\Omega(\mathbf{p},x)}.$$

We wish to prove that $\varphi(\mathbf{p}) \succsim \varphi(\mathbf{q})$, that is, $(\langle \mathbf{p}\rangle, f_{\mathbf{p}}) \succsim (\langle \mathbf{q}\rangle, f_{\mathbf{q}})$. Let us enumerate $\mathrm{supp}(\mathbf{p})$ as $\{x_1, ..., x_k\}$. Then, as $\mathbf{q}$ is a mean-preserving spread of $\mathbf{p}$, there exist simple lotteries $\theta_1, ..., \theta_k$ on X such that

$$\mathbf{q} = \sum_{i=1}^{k} \mathbf{p}(x_i)\theta_i \quad \text{and} \quad \mathbb{E}(u, \theta_i) = x_i \text{ for each } i = 1, ..., k.$$

We let $C_i := \mathrm{supp}(\theta_i)$ for each $i = 1, ..., k$. In view of the continuity of $\succsim$, it is without loss of generality to presume that these sets are disjoint, for otherwise the condition can be ensured by infinitesimal displacements of the involved support points. Then, $\mathrm{supp}(\mathbf{q}) = C_1 \cup \cdots \cup C_k$ and $\mathbf{q}(C_1) = \mathbf{p}(x_1)$. Applying (27) with $C = C_1$ and $f = f_{\mathbf{q}}$, therefore, we get

$$\left(\langle \mathbf{q}\rangle, \theta_1 \mathbf{1}_{\Omega(\mathbf{q},C_1)} + f_{\mathbf{q}} \mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},C_1)}\right) \succsim (\langle \mathbf{q}\rangle, f_{\mathbf{q}}).$$

But we also have $\sum_{x \in C_2} \frac{\mathbf{q}(x)}{\mathbf{q}(C_2)} f_{\mathbf{q}}(x) = \theta_2$, so applying (27) with $C = C_2$ and $f = \theta_1 \mathbf{1}_{\Omega(\mathbf{q},C_1)} + f_{\mathbf{q}} \mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},C_1)}$ yields

$$\begin{aligned} \left(\langle \mathbf{q}\rangle, \theta_1 \mathbf{1}_{\Omega(\mathbf{q},C_1)} + \theta_2 \mathbf{1}_{\Omega(\mathbf{q},C_2)} + f_{\mathbf{q}} \mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},C_1 \cup C_2)}\right) &\succsim \left(\langle \mathbf{q}\rangle, \theta_1 \mathbf{1}_{\Omega(\mathbf{q},C_1)} + f_{\mathbf{q}} \mathbf{1}_{\Omega\backslash\Omega(\mathbf{q},C_1)}\right) \\ &\succsim (\langle \mathbf{q}\rangle, f_{\mathbf{q}}). \end{aligned}$$

Then, continuing this way inductively, we find:

$$\left(\langle \mathbf{q}\rangle, \sum_{i=1}^{k} \theta_i \mathbf{1}_{\Omega(\mathbf{q},C_i)}\right) \succsim (\langle \mathbf{q}\rangle, f_{\mathbf{q}}). \tag{28}$$

Next, we observe that $(\langle \theta_i\rangle, \theta_i \mathbf{1}_\Omega) \sim (\langle \theta_i\rangle, \delta_{\mathbb{E}(\theta_i)} \mathbf{1}_\Omega) \sim (\langle \mathbf{q}\rangle, \delta_{\mathbb{E}(\theta_i)} \mathbf{1}_\Omega) = (\langle \mathbf{q}\rangle, \delta_{x_i} \mathbf{1}_\Omega)$ for each $i = 1, \dots, k$ by (9) and the State Invariance Axiom, respectively. By monotonicity, this implies

$$\left(\langle \mathbf{q}\rangle, \sum_{i=1}^{k} \theta_i \mathbf{1}_{\Omega(\mathbf{q},C_i)}\right) \sim \left(\langle \mathbf{q}\rangle, \sum_{i=1}^{k} \delta_{x_i} \mathbf{1}_{\Omega(\mathbf{q},C_i)}\right),$$

while, by the State Invariance Axiom, we have

$$\left(\langle \mathbf{q}\rangle, \sum_{i=1}^{k} \delta_{x_i} \mathbf{1}_{\Omega(\mathbf{q},C_i)}\right) \sim (\langle \mathbf{p}\rangle, f_{\mathbf{p}}).$$

Combining these observations with (28), we conclude that $(\langle \mathbf{p}\rangle, f_{\mathbf{p}}) \succsim (\langle \mathbf{q}\rangle, f_{\mathbf{q}})$. Theorem 9 is now proved.

COMPLETION OF THE PROOF OF PROPOSITION 7

We now turn to the proof of the "only if" part of Proposition 7. Suppose that $\trianglerighteq$ is $\mathcal{S}$-pessimistic, and define $\succsim$ as in the proof of Proposition 7 above. Then, Lemma A.2 applies to $\succsim$, so the proof of Theorem 9 goes through *verbatim*, establishing that $\trianglerighteq$ is risk averse. Proposition 7 is now proved.

PROOF OF THEOREM 11

Let $\trianglerighteq$ be an $\mathcal{S}$-pessimistic preference relation in $\mathfrak{R}(X)$, and recall that there is a unique $\succsim \in \mathcal{S}$ with $\trianglerighteq = \Phi(\succsim)$. Take any positive integer n, and assets $x_1, ..., x_n$ on X with $x_1 \simeq \cdots \simeq x_n$. By definition, the latter condition means that $\mathbf{p}_{x_1} \simeq \cdots \simeq \mathbf{p}_{x_n}$, so

$$\varphi(\mathbf{p}_{x_1}) \sim \cdots \sim \varphi(\mathbf{p}_{x_n}). \tag{29}$$

Now, fix any nonnegative numbers $\lambda_1, ..., \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$. To simplify our notation, we put $x_\lambda := \lambda_1 x_1 + \cdots + \lambda_n x_n$, which is itself an asset on X. We wish to show that $\mathbf{p}_{x_\lambda} \trianglerighteq \mathbf{p}_{x_1}$.

Define $\mathbf{x} := (x_1, ..., x_n)$, which is an X^n-valued random variable on $([0,1], \mathcal{B}, \ell)$ with finite range. Put

$$Q := \{\{\mathbf{x} = a\} : a \in \text{ rng}(\mathbf{x})\}.$$

(Here, rng($\mathbf{x}$) stands for the range of $\mathbf{x}$.) Pick any map $\psi : Q \to X$, and define

$$\mathbf{p} := \sum_{a \in \text{rng}(\mathbf{x})} \ell(\{\mathbf{x} = a\}) \delta_{\psi(\{\mathbf{x}=a\})},$$

which is a simple lottery on X. Finally, we define the acts f_i and f_λ in $\mathcal{F}(\mathbf{p})$ by

$$f_i := \sum_{a \in \text{rng}(\mathbf{x})} \delta_{a_i} \mathbf{1}_{\Omega(\mathbf{p}, \psi(\{\mathbf{x}=a\}))} \quad \text{and} \quad f_\lambda := \sum_{a \in \text{rng}(\mathbf{x})} \delta_{\lambda_1 a_1 + \cdots + \lambda_n a_n} \mathbf{1}_{\Omega(\mathbf{p}, \psi(\{\mathbf{x}=a\}))},$$

for each $i = 1, ..., n$. (Here, for any real n-vector a, we denote by a_i the ith component of a.)

Let us first note that

$$(\langle \mathbf{p} \rangle, f_i) \sim \varphi(\mathbf{p}_{x_i}) \quad \text{for each } i = 1, ..., n, \lambda. \tag{30}$$

To see this, fix any i in $\{1, ..., n\}$, and define $\sigma : \Omega(\mathbf{p}) \to \Omega(\mathbf{p}_{x_i})$ by

$$\sigma(\Omega(\mathbf{p}, \psi(\{\mathbf{x} = a\}))) := \Omega(\mathbf{p}_{x_i}, a_i).$$

Then, for any a in rng($\mathbf{x}$), we have $\mathbf{p}(\sigma^{-1}(\Omega(\mathbf{p}_{x_i}, a_i))) = \ell(\{x_i = a_i\}) = \mathbf{p}_{x_i}(\Omega(\mathbf{p}_{x_i}, a_i))$ and $f_i(\omega) = \delta_{a_i} = f_{\mathbf{p}_{x_i}}(\omega')$ for every $\omega \in \Omega(\mathbf{p}, \psi(\{\mathbf{x} = a\}))$ and $\omega' \in \Omega(\mathbf{p}_{x_i}, a_i)$. By the State Invariance Axiom, therefore, (30) follows.

Next, we note that

$$f_\lambda \sim^{\mathbf{p}} \lambda_1 f_1 + \cdots + \lambda_n f_n. \tag{31}$$

Indeed, by (9), we have $(\lambda_1 \delta_{a_1} + \cdots + \lambda_n \delta_{a_n}) \mathbf{1}_\Omega \sim^{\mathbf{p}} \delta_{\lambda_1 a_1 + \cdots + \lambda_n a_n} \mathbf{1}_\Omega$ for every $a \in \text{rng}(\mathbf{x})$, and hence, by monotonicity of $\succsim^{\mathbf{p}}$,

$$\sum_{a \in \text{rng}(\mathbf{x})} \delta_{\lambda_1 a_1 + \cdots + \lambda_n a_n} \mathbf{1}_{\Omega(\mathbf{p}, \psi(\{\mathbf{x}=a\}))} \sim^{\mathbf{p}} \sum_{a \in \text{rng}(\mathbf{x})} \left(\sum_{i=1}^n \lambda_i \delta_{a_i} \right) \mathbf{1}_{\Omega(\mathbf{p}, \psi(\{\mathbf{x}=a\}))} = \sum_{i=1}^n \sum_{a \in \text{rng}(\mathbf{x})} \lambda_i \delta_{a_i} \mathbf{1}_{\Omega(\mathbf{p}, \psi(\{\mathbf{x}=a\}))}.$$

which yields (31).

We are now ready to complete our proof. By (29) and (30), we have $f_1 \sim^{\mathbf{p}} \cdots \sim^{\mathbf{p}} f_n$. But, as $\trianglerighteq$ is $\mathcal{S}$-pessimistic, $\succsim$ is uncertainty averse. In particular, $\succsim^{\mathbf{p}}$ satisfies the Uncertainty Aversion axiom. Applying this axiom inductively, then, $\lambda_1 f_1 + \cdots + \lambda_n f_n \succsim^{\mathbf{p}} f_1$. In view of (31), therefore, we find that $f_\lambda \succsim^{\mathbf{p}} f_1$. Using (30) one more time, then, $\varphi(\mathbf{p}_{x_\lambda}) \succsim \varphi(\mathbf{p}_{x_1})$, that is, $\mathbf{p}_{x_\lambda} \trianglerighteq \mathbf{p}_{x_1}$. Theorem 11 is now proved.

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